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1,2 Conjecture

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1, 2 Conjecture

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Abstract

Let us assign weights to the edges and vertices of a simple graph G. As a result we obtain a vertex-colouring of G by sums of weights assigned to the vertex and its adjacent edges. Can we receive a proper coloring using only weights 1 and 2 for an arbitrary G?

We give a positive answer for bipartite and complete graphs and for the ones with $\Delta(G) \leqslant 3$.

1 Introduction

A *k-total-weighting* of a simple graph G is an assignment of an integer weight, $w(e), w(v) \in \{1, \ldots, k\}$ to each edge e and each vertex v of G. The k-total-weighting is *neighbour-distinguishing* (or vertex colouring, see [1]) if for every edge uv, $w(u) + \sum_{e \ni u} w(e) \neq w(v) + \sum_{e \ni v} w(e)$. In such a case we say that G *permits* a neighbour-distinguishing k-total-weighting. The smallest k for which G permits a neighbour-distinguishing k-total-weighting we denote by $\tau(G)$.

Similar parameter, but in the case of an *edge* (not total) weighting was introduced and studied in [2] by Karoński, Łuczak and Thomason. They asked if each, except for a single edge, simple connected graph permits a *neighbour-distinguishing* 3-*edge-weighting*, and showed that this statement holds e.g. for 3-colourable graphs. It is also known, see [1], that each *nice* (not containing a connected component which has only one edge) graph permits a neighbour-distinguishing 16-edge-weighting, hence the considered parameter is finite.

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Note that if a graph permits a neighbour-distinguishing k-edge-weighting, then it also permits a neighbour-distinguishing k-total-weighting (it is enough to put ones at all vertices), hence we obtain an upper bound $\tau(G) \leqslant 16$ for all graphs and $\tau(G) \leqslant 3$ for 3-colourable graphs (for all graphs if the conjecture of Karoński, Łuczak and Thomason holds). Therefore, we formulate the following conjecture.

Conjecture 1 *Every simple graph permits a neighbour-distinguishing* 2*-total-weighting.*

It might seem quite plausible in the face of the result of Addarrio-Berry, Dalal and Reed from [1], which say that for any fixed $p \in (0, 1)$ the random graph $G_{n,p}$ asymptotically almost surely permits neighbour-distinguishing 2-edge-weighting. In the following section we shall show that Conjecture 1 holds for bipartite and complete graphs and for graphs with $\Delta(G) \leqslant 3$, see Theorem 7.

It is also worth mentioning here that our reasonings correspond with the recent study of Bača, Jendrol, Miller and Ryan. In [3] they introduced and studied a parameter called *total vertex irregularity strength*, which is the smallest k for which there exists a k-total-weighting such that each vertex of a graph receive a different colour, i.e. $w(u) + \sum_{e \ni u} w(e) \neq w(v) + \sum_{e \ni v} w(e)$ for each (not only neighbouring ones) pair of vertices u, v. This parameter, as well as the other parameters mentioned in this section, may be viewed as descendants of the well known *irregularity strength* of a graph, see [4].

2 Results

Our aim is to show that $\tau(G) \leqslant 2$ for a graph G. Note then first that $\tau(G) = 1$ iff each two neighbours have different degrees in G. Since we wish to distinguish only neighbours, we may assume that G is a simple *connected* graph. Let for a given total-weighting w of G, $c_w(v) := w(v) + \sum_{e \ni v} w(e)$ (or $c(v)$ for short if the weighting w is obvious). For the convenience of the notation we shall call w a labelling (of the vertices and edges) and c_w (or c) a weighting of the vertices of G in what follows. Surprisingly easily we may prove the following statement.

Observation 2 $\tau(G) \leqslant 2$ *for bipartite graphs.*

Proof. Let us first arbitrarily label the edges of G using 1 or 2. Then put 1 or 2 at vertices so that the resulting weights of the vertices in one colour class are even and odd in the other one. ■

Note that $\tau(G) = 2$ if G is a single edge, hence our parameter makes sense for all, not only nice (as it was in the case of edge-weighting), graphs.

Though the following observation is the consequence of [3], we present here our proof for the cohesion of the article.

Observation 3 $\tau(G) = 2$ *for complete graphs.*

Proof. For K_2 it is enough to put 1 on the edge and different numbers, namely 1 and 2, at vertices. This way, the weights of vertices equal 2 and 3. Then we use induction to show that we can always label K_n using 1 and 2 so that its vertices obtain weights being n consecutive integers.

Assume we have already labelled a graph K_{n-1} in the described way and let us add a new vertex v joining it by a single edge with each vertex of K_{n-1}. Notice that the vertices of K_{n-1} obtained weights from the interval $[n-1, 2n-2]$. If the greatest of them equals $2n-3$, we put twos at v and on all the edges incident with it. This way, the vertices of K_n obtain n different weights from the interval $[n+1, 2n]$. Analogously, if the greatest weight at a vertex of K_{n-1} equals $2n-2$, we put ones at vertex v and all the edges incident with it. ■

Lemma 4 $\tau(G) = 2$ *for cycles (hence also for 2-regular graphs).*

Proof. To label an even cycle it is enough to put ones on all the edges and then alternately ones and twos at vertices along the cycle.

Since $\tau(C_3) = 2$ by Observation 3, we may assume G is an odd cycle of size at least 5 with v, u, w being consecutive vertices on this cycle. Then create an even cycle G' of G by removing u and adding an edge vw. Such a graph we label as described in the previous paragraph. Then we delete the edge vw and exchange the labels of v and w, and finally put twos at u and on the edges incident with it. It is easy to verify that the resulting labelling complies with our requirements. ■

Lemma 5 $\tau(G) = 2$ *for cubic graphs.*

Proof. Let G be a connected cubic graph. If $G = K_4$, then we are done by Observation 3, hence we may assume $G \neq K_4$. By Brooks's theorem $\chi(G) \leq 3$. If $\chi(G) = 2$, then G is bipartite and the statement follows by Observation 2. So, let us consider the case $\chi(G) = 3$. Denote by A, B and C the colour classes of G. Without loss of generality we may assume that A is as large as possible, and, subject to the choice of A, B is also as large as possible. This implies, in particular, that each vertex from $B \cup C$ has at least one neighbour in A and that each vertex from C has at least one neighbour in B. We define a labelling w in the following way. First, we label the edges between A and $B \cup C$ by 2 and the edges between B and C by 1. Next we label the vertices.

All the vertices from A get label 2. This way, all the vertices from A have total weights equal to 8. For each vertex belonging to $B \cup C$ we then choose a label 1 or 2 in such a way that the total weights of the vertices from B are odd and the total weights of vertices from C are even. Since each vertex from C is incident with at least one edge labelled with 1, their total weights cannot exceed 7, so, these total weights are at most 6. Therefore, the above procedure gives a labelling with total weight distinct for vertices belonging to distinct sets of partition. ■

Remark 6 *Analogous reasoning results in conclusion that $\tau(G) = 2$ for all regular tripartite graphs.*

Theorem 7 *$\tau(G) \leqslant 2$ for all graphs with $\Delta(G) \leqslant 3$.*

Proof. The theorem holds for a single edge, thus we argue by induction on the number of edges of G, where G is connected.

If $\delta(G) = 3$, then G is a cubic graph and we are done by Lemma 5.

If $\delta(G) = 1$ and $N_G(v) = \{u\}$, we label a graph $G - v$ by induction and delete the label of u. Notice that using labels 1 or 2 at u and on vu we may add 2, 3 or 4 to the total weight of u. Therefore, since u has at most two neighbours different from v in G, we easily differentiate u from them by putting 1 or 2 at u and on vu. Then we complete the labelling of G by putting 1 or 2 at v, so that the weights at v and u are different.

The case $\delta(G) = 2 = \Delta(G)$ was discussed in Lemma 4.

Therefore, we may assume $\delta(G) = 2$ and $\Delta(G) = 3$. A sequence $v_0, v_1, \ldots, v_n$ ($n \geqslant 2$) of vertices of G we shall call a *suspended trail* of length n iff $v_{i-1}v_i$ are edges of G for $i = 1, \ldots, n$, $d_G(v_0) = 3 = d_G(v_n)$ and $d_G(v_j) = 2$ for $0 < j < n$ (notice, we do not require v_0 and v_n to be distinct). Let $v_0, v_1, \ldots, v_n$ be the longest suspended trail in G. Assume first its length is at least four and $v_0 \neq v_n$ or is at least five and $v_0 = v_n$. In such a case, if we remove v_1, v_2 (hence also three edges) from G and add an edge v_0v_3, then $v_0, v_3, v_4, \ldots, v_n$ will be a suspended trail in the resulting graph G'. We may label then G' by induction and extend this labelling to G. First remove v_0v_3 and put $w(v_0v_1) = w(v_0v_3)$, $w(v_1v_2) = w(v_3v_4)$, $w(v_2v_3) = w(v_0v_3)$, $w(v_1) = w(v_3)$. This way the total weights of v_1 and v_3 are the same as the weight of v_3 in G', and it is easy to complete the labelling by putting 1 or 2 at v_2 so that its weight is different from the weight of v_1 (and v_3). Therefore, we may assume the length of the suspended trail is quite small, hence we distinguish the following six cases.

Case 1: $n = 4$ and $v_0 = v_n$. Then we remove v_1, v_2, v_3 from G and label the resulting graph G' by induction. Then we label the edges $v_{i-1}v_i$, i=1,2,3,4, with ones. Then we change (if necessary) the label of v_0 so that its weight is different from the weight of its only neighbour from G'. Subsequently, we label v_1 and v_3 with the same number so that their weights are different from the weight of v_0. Since the weights of v_1 and v_3 are the same, we easily choose the label for v_2.

Case 2: $n = 3$ and $v_0 = v_n$. Analogously, we remove v_1, v_2 from G and label the resulting graph G' by induction. Then we put $w(v_0v_1) = 1$, $w(v_1) = 1$, $w(v_1v_2) = 1$, $w(v_2) = 1$ and $w(v_2v_0) = 2$. Then we change (if necessary) the label of v_0 so that its weight is different from the weight of its only neighbour from G'. Since then $c(v_1) = 3$, $c(v_2) = 4$ and $c(v_0) \geqslant 5$, this labelling is neighbour-distinguishing.

Case 3: $n = 2$, $v_0 \neq v_n$ and $v_0v_n \in E(G)$. Then we remove v_1 and v_0v_2 from G and label the resulting graph G' by induction (though G' may not be connected, we can label each of its connected components independently). Then we put $w(v_1) = 1$, $w(v_1v_2) = 1$, $w(v_2v_0) = 2$ and relabel v_2 (if necessary) so that its weight is different from the weight of its only neighbour from G'. Then we label v_0 and v_0v_1 so that the weight of v_0 is different from the weights of its only neighbour from G' and v_2. By our construction $c(v_0), c(v_2) \geqslant 5$ and $c(v_1) \leqslant 4$, hence this labelling is neighbour-distinguishing.

Case 4: $n = 3$, $v_0 \neq v_n$ and $v_0v_n \in E(G)$. Analogously, we remove v_1, v_2 and v_0v_3 from G and label the resulting graph G' by induction. Then we put $w(v_1v_2) = 1$, $w(v_2) = 1$, $w(v_2v_3) = 1$, $w(v_3v_0) = 2$ and relabel v_3 (if necessary) so that its weight is different from the the weight of its only neighbour from G'. Then we label v_0 and v_0v_1 so that the weight of v_0 is different from the weights of its only neighbour from G' and v_3. By our construction $c(v_0), c(v_3) \geqslant 5$ and $c(v_2) = 3$, hence it is enough to put 1 or 2 at v_1, so that $c(v_1) = 4$.

Case 5: $n = 2$, $v_0 \neq v_n$ and $v_0v_n \notin E(G)$. Then we remove v_1 from G and add an edge v_0v_2. The resulting graph G' (it may be a cubic graph) we label by induction. If $w(v_0v_2) = 1$, then we remove the edge v_0v_2 and put ones on v_0v_1, v_1v_2 and at v_1. This way the weights of v_0 and v_2 remain unchanged and are greater than three, while $c(v_1) = 3$. Therefore, we may assume $w(v_0v_2) = 2$, $w(v_0) = a$ and $w(v_2) = b$. Then we remove the edge v_0v_2, put $w(v_0v_1) = a$, $w(v_1v_2) = b$ and change the labels at v_0 and v_2 to twos. This way, the weights of v_0 and v_2 remain as they were in G'. Finally, we put one at v_1 and obtain $c(v_1) = a + b + 1$, $c(v_0) \geqslant 2 + a + 2$ and $c(v_2) \geqslant 2 + b + 2$. Since $a, b \leqslant 2$, we have $c(v_1) < c(v_0)$ and $c(v_1) < c(v_2)$.

Case 6: $n = 3$, $v_0 \neq v_n$ and $v_0v_n \notin E(G)$. Then we remove v_1, v_2 from G and add an edge v_0v_3. The resulting graph G' we label by induction. Since v_0 and v_3 are neighbours in G', their weights are different, hence the weight of one of them must exceed four. Assume then $c(v_3) \geqslant 5$. If $w(v_0v_3) = 1$, then we remove the edge v_0v_3 and put $w(v_0v_1) = w(v_1) = w(v_1v_2) = w(v_2v_3) = 1$, $w(v_2) = 2$. This way the weights of v_0 and v_3 remain unchanged, hence $c(v_0) \geqslant 4$ and $c(v_3) \geqslant 5$, while $c(v_1) = 3$ and $c(v_2) = 4$. Therefore, we may assume $w(v_0v_3) = 2$, $w(v_0) = a$ and $w(v_3) = b$. Moreover, analogously as above, we may assume $c(v_0) \geqslant 5$ and $c(v_3) \geqslant 6$ (since v_0 and v_3 are neighbours in G'). Then we remove the edge v_0v_3, put $w(v_0v_1) = a$, $w(v_2v_3) = b$ and change the labels at v_0 and v_3 to twos. This way, the weights of v_0 and v_3 remain as they were in G'. Then we put ones at v_1 and on v_1v_2, and obtain $c(v_1) = a + 1 + 1 \leqslant 4 < c(v_0)$. Then we put $d \in \{1, 2\}$ at v_2, so that its weight is different from the weight of v_1. Consequently, we have $c(v_2) = 1 + d + b \leqslant 5 < c(v_3)$, what finishes the proof. ■

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