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The Distinguishing Index of the Cartesian Product of Finite Graphs*

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Abstract

The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that is only preserved by the identity automorphism. In this paper we investigate the distinguishing index of the Cartesian product of connected finite graphs. We prove that for every $k \geq 2$, the k-th Cartesian power of a connected graph G has the distinguishing index equal to two with the only exception $D'(K_2^2) = 3$. We also prove that if G and H are connected graphs that satisfy the relation $2 \leq |G| \leq |H| \leq 2^{|G|}\left(2^{\|G\|} - 1\right) - |G| + 2$, then $D'(G \Box H) \leq 2$ unless $G \Box H = K_2^2$.

Keywords: edge colouring; symmetry breaking; distinguishing index; Cartesian product

Mathematics Subject Classifications: 05C15, 05E18

1 Introduction

We use the standard graph theory notation (cp. [6]). In particular, $\mathrm{Aut}(G)$ denotes the automorphism group of a graph G.

An edge colouring *breaks an automorphism* $\varphi \in Aut(G)$ if φ does not preserve this colouring, i.e., there exists an edge of G that is mapped by φ

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to an edge coloured differently. The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that breaks all non-trivial automorphisms of G. Such a d-colouring is called *distinguishing*. This notion was introduced by Kalinowski and Pilśniak [9] as an analogue of the well-known *distinguishing number* $D(G)$ of a graph G defined by Albertson and Collins [2] for vertex colourings. Obviously, the distinguishing index is not defined for K_2, thus from now on, we assume that K_2 is not a connected component of any graph being considered.

The distinguishing index of some examples of graphs was exhibited in [9]. For instance, $D'(P_n) = D(P_n) = 2, n \geq 3$; $D'(C_n) = D(C_n) = 2, n \geq 6$, and $D'(C_n) = 3, n = 3, 4, 5$. There exist graphs G for which $D'(G) < D(G)$, for instance $D'(K_n) = D'(K_{p,p}) = 2$, for any $n \geq 6$ and $p \geq 4$ while $D(K_n) = n$ and $D(K_{p,p}) = p + 1$. It is also possible that $D'(G) > D(G)$, and all trees satisfying this inequality were characterized in [9]. A general upper bound of the distinguishing index was proved in [9].

Theorem 1 *[9] If G is a finite connected graph of order $n \geq 3$, then $D'(G) \leq D(G) + 1$. Moreover, if Δ is the maximum degree of G, then $D'(G) \leq \Delta$ unless G is a C_3, C_4 or C_5.*

The distinguishing index was already investigated also for infinite graphs [3] and their Cartesian product [4].

In this paper we aim to present some results for the distinguishing index of the Cartesian powers and the Cartesian product of connected graphs.

The Cartesian product of graphs G and H is a graph denoted $G \Box H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. We denote $G \Box G$ by G^2, and we recursively define the *k-th Cartesian power* of G as $G^k = G \Box G^{k-1}$.

A graph G is called *prime* if $G = G_1 \Box G_2$ implies that either G_1 or G_2 is K_1. It has been proved by Sabidussi and Vizing (cp. [6]) that every graph has a prime factor decomposition with respect to the Cartesian product. Moreover, for connected graphs this decomposition is unique up to the order of isomorphic factors. Two graphs G and H are called *relatively prime* if K_1 is the only common factor of G and H.

Let v be a vertex of H. A *G^v-layer* (called also a *horizontal layer* of $G \Box H$) is the subgraph induced by the vertex set $\{(u, v) : u \in V(G)\}$. An

H^u-*layer*, called *vertical*, is defined analogously for a vertex u of G. Clearly, each horizontal layer is isomorphic to G and each vertical one is isomorphic to H. Therefore, speaking of a specified layer of $G\Box H$, we shall usually use only one coordinate of a vertex. The same refers to edges.

The automorphism group of the Cartesian product was characterized by Imrich [7], and independently by Miller [11].

Theorem 2 *[7], [11] Suppose ψ is an automorphism of a connected graph G with prime factor decomposition $G = G_1\Box G_2\Box \ldots \Box G_r$. Then there is a permutation π of the set $\{1, 2, \ldots, r\}$ and there are isomorphisms $\psi_i \colon G_{\pi(i)} \mapsto G_i$, $i = 1, \ldots, r$, such that*

$$\psi(x_1, x_2, \ldots, x_r) = (\psi_1(x_{\pi(1)}), \psi_2(x_{\pi(2)}), \ldots, \psi_r(x_{\pi(r)})).$$

It follows in particular that every automorphism of the Cartesian product of two relatively prime graphs is a composition of a permutation of vertical layers generated by an automorphism of G and a permutation of horizontal layers generated by an automorphism of H. For more about the Cartesian product, consult [6].

The distinguishing number of the Cartesian powers of graphs has been thoroughly investigated. It was first proved by Albertson [1] that if G is a connected prime graph, then $D(G^k) = 2$ whenever $k \geq 4$, and if $|V(G)| \geq 5$, then also $D(G^3) = 2$. Next, Klavžar and Zhu [10] showed that for any connected graph G with a prime factor of order at least 3 the distinguishing number $D(G^k) = 2$ for $k \geq 3$. Both results were obtained using the Motion Lemma [14]. Finally, Imrich and Klažar [8] solved the problem completely by the following result, and their proof is not based on the Motion Lemma but rather on the algebraic properties of the automorphism group of the Cartesian product of graphs.

Theorem 3 *[8] Let G be a connected graph and $k \geq 2$. Then $D(G^k) = 2$ except for the graphs K_2^2, K_2^3, K_3^2 whose distinguishing number is three.*

In Section 3 we obtain an analogous result (Theorem 16) for the distinguishing index. Earlier, in Section 2 we prove some lemmas and propositions used then in proofs of results in the next sections. Some of them are of interest as such.

In the same paper [8], Imrich and Klažar obtained a sufficient condition when the distinguishing number of the Cartesian product of two relatively prime graphs equals two.

Theorem 4 *[8] Let G and H be connected, relatively prime graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $D(G\Box H) \leq 2$.

Recently, this result was extended by Estaji, Imrich, Kalinowski and Pilśniak in [5] for graphs which are not necessarily relatively prime.

Theorem 5 *[5] Let G and H be connected graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $D(G\Box H) \leq 2$ unless $G\Box H \in \{K_2^2, K_2^3, K_3^2\}$.

In Section 4 we prove an analogous result (Theorem 19) for the distinguishing index of the Cartesian product of connected graphs. We also obtain a slightly stronger result for trees (Theorem 17).

In proofs, we usually use colours $1, \ldots, d$. If $d = 2$, then we also use colours 0 and 1, or alternatively red and blue.

2 Examples and lemmas

The Cartesian product $P_m\Box P_n$ of two paths of orders m and n, respectively, has the distinguishing index equal to 2, unless $m = n = 2$ since then $D'(P_2\Box P_2) = D'(C_4) = 3$. Indeed, it suffices to colour differently one edge of the P_n-layer containing the vertex (u, v), where u and v are the end-vertices of P_m and P_n, respectively.

For the Cartesian product of a cycle C_n with a path P_m we also have $D'(P_m\Box C_n) = 2$. It is easy to see that it suffices to colour two adjacent edges, one in a C_n-layer and one in a P_m-layer, red and all other edges blue.

The Cartesian product of two cycles C_n and C_m also has the distinguishing index equal to two. When $m \neq n$, the same colouring as in the case $P_m\Box C_n$ breaks all non-trivial automorphisms. When $m = n$, we additionally colour red a third edge such that these three red edges form a path of length three (see Figure 1 for $m = n = 3$). It is worth noting that these results do not depend on the relation between n and m.

The factors considered above have small distinguishing indices. Cycles C_3, C_4, C_5 have the distinguishing index equal to three and paths (except

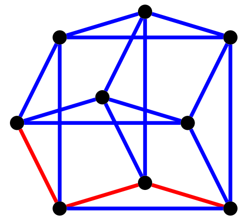

Figure 1: A distinguishing 2-colouring of $C_3 \Box C_3$

P_2) as well as cycles C_n, for $n \geq 6$, have the distinguishing index equal to two. The problem becomes more complicated when we consider factors with higher distinguishing index. An extremal example of a graph with the largest distinguishing index compared to its size is a star $K_{1,n}$ with $n \geq 2$ since $D'(K_{1,n}) = n = \|K_{1,n}\|$. But first let us state the following result concerning the Cartesian product $H \Box K_2$ called a *prism* of a graph H. We shall make use of it in next parts of the paper.

Proposition 6 *If H is a graph with $D'(H) = d \geq 2$, then $D'(H \Box K_2) \leq d$. Moreover, if $D'(H) = 2$, then there holds the equality $D'(H \Box K_2) = 2$.*

Proof. We colour the edges of one H-layer with a distinguishing d-colouring, and all the edges of the other H-layer with the same colour, say 1. Next, we colour all edges of K_2-layers with colour 2. Thus all automorphisms of the Cartesian product $H \Box K_2$ generated by the automorphisms of H are broken, since one of the H-layers assumes a distinguishing colouring. Also, the two H-layers cannot be interchanged as they have different numbers of edges coloured with 1.

If H has a factor H' isomorphic to K_2, then $K_2 \Box H$ has an automorphism interchanging H' with K_2. However, since all K_2-layers have only colour 2 and there exists an H-layer with all edges coloured with 1, such an automorphism does not preserve the colouring.

The equality for $d = 2$ is obvious since the prism of every graph has a non-trivial automorphism. $\Box$

It may happen that $D'(H \Box K_2) < d$ when $D'(H) = d \geq 3$. For instance, we showed above that $D'(C_m \Box P_2) = 2$ while $D'(C_m) = 3$ for $m = 3, 4, 5$.

Proposition 7 *If $m \geq 2$ and $n \geq 2$, then*

$$D'(K_{1,n} \Box P_m) = \left\lceil \sqrt[2m-1]{n} \right\rceil$$

unless $m = 2$ *and* $n = r^3$ *for some integer* r. *In the latter case* $D'(K_{1,n} \square P_2) = \sqrt[3]{n} + 1$.

Proof. Let d be a positive integer such that $(d-1)^{2m-1} < n \leq d^{2m-1}$. Denote by v the central vertex of the star $K_{1,n}$, by $v_1, \ldots, v_n$ its pendant vertices, and by $u_1, \ldots, u_m$ consecutive vertices of P_m.

Suppose first that $m \geq 3$. Clearly, every automorphism of $K_{1,n} \square P_m$ maps the P_m^v-layer onto itself. We colour the first edge of this layer with 1 and all other edges of it with 2. Thus the P_m^v-layer is fixed by every automorphism, hence no $K_{1,n}$-layers can be permuted.

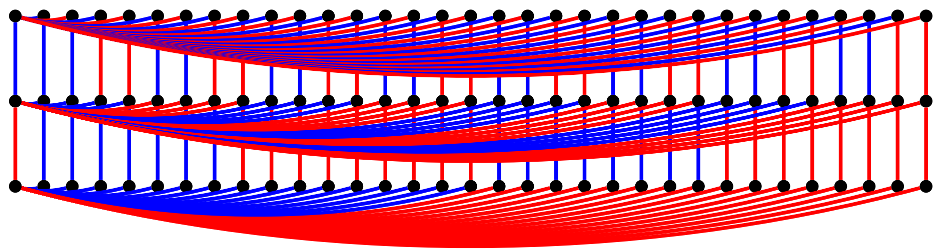

Figure 2: A distinguishing 2-colouring of $K_{1,32} \square P_3$

We want to show that the remaining edges of $K_{1,n} \square P_m$ can be coloured in such a way that P_m-layers also cannot be interchanged. Then all non-trivial automorphisms of $K_{1,n} \square P_m$ will be broken. Note that a colouring of all edges yet uncoloured can be fully described by defining a matrix M with $2m-1$ rows and n columns such that in the j-th column the initial $m-1$ elements are colours of consecutive edges of the $P_m^{v_j}$-layer, and the other m elements are colours of the edges of $K_{1,n}$-layers incident to consecutive vertices of the $P_m^{v_j}$-layer. It is easily seen that there exists a permutation of P_m-layers preserving colours if and only if M contains at least two identical columns. There are exactly d^{2m-1} sequences of length $2m-1$ with elements from the set $\{1, \ldots, d\}$, hence there exists a colouring with d colours such that every column of M is distinct. Therefore, $D'(K_{1,n} \square P_m) \leq d = \lceil \sqrt[2m-1]{n} \rceil$. On other hand, $n > (d-1)^{2m-1}$ so for every edge $(d-1)$-colouring of $K_{1,n} \square P_m$, the corresponding matrix has to contain two identical columns, therefore $D'(K_{1,n} \square P_m) > d-1$. Figure 2 presents the case $n = 32$ and $m = 3$.

For $m = 2$, we colour the edges of $K_{1,n} \square P_2$ in the same way. The only difference is that every P_2-layer has only one edge, hence the two $K_{1,n}$-layers need not be fixed. This is the case when $n = d^3$, because then each element of $\{1, \dots, d\}^3$ is a column in M, and there exists a permutation of columns of M which together with the transposition of rows of M defines a non-trivial automorphism of $K_{1,n} \Box P_2$ preserving the colouring. Thus we need an additional colour for one edge in a $K_{1,n}$-layer. When $n < d^3$, we put the sequence $(1, 1, 2)$ as the first column of M, and we do not use the sequence $(1, 2, 1)$ any more, thus breaking the transposition of the $K_{1,n}$-layers, and all automorphisms of $K_{1,n} \Box P_2$. $\Box$

Observe that the case $n = 1$ is covered by Proposition 6.

Proposition 8 *If $m \geq 3$ and $n \geq 2$, then*

$$D'(K_{1,n} \Box C_m) = \lceil \sqrt[2m]{n} \rceil$$

unless $m \leq 5$ and $n = 2^{2m}$. In the latter case $D'(K_{1,n} \Box C_m) = \sqrt[2m]{n} + 1 = 3$.

Proof. Let d be a positive integer such that $(d-1)^{2m} < n \leq d^{2m}$. Clearly, the C_m^v-layer, where v is a central vertex of $K_{1,n}$, is mapped onto itself by every automorphism of $K_{1,n} \Box C_m$. The idea of the proof is the same as that of Proposition 7 but here the matrix M of colours $1, \dots, d$ has $2m$ rows as C_m has one more edge than P_m.

Assume first that $m \geq 6$. We put a distinguishing 2-colouring of the C_m^v-layer. Thus all permutations of $K_{1,n}$-layers are broken. To break permutations of C_m-layers, it suffices to ensure the columns of M are pairwise distinct. This is clearly possible since $n \leq d^{2m}$. If we used less number of colours, then M would have two equal columns because the number n of columns is greater than $(d-1)^{2m}$. Hence, $D'(K_{1,n} \Box C_m) = \lceil \sqrt[2m]{n} \rceil$ in this case.

Now, let $3 \leq m \leq 5$. If $d \geq 3$, then we put a distinguishing 3-colouring of the C_m^v-layer, and we argue as in the previous case. Then suppose that $2 \leq n \leq 2^{2m}$. It is easy to check that for each m there exists an edge 2-colouring c of the cycle C_m that is preserved by a unique $\varphi_0 \in \mathrm{Aut}(C_m) \setminus \{\mathrm{id}\}$, and there exists another 2-colouring c' such that there does not exist a non-trivial automorphism of C_m that preserves both colourings. We put the colouring c on the C_m^v-layer and the colouring c' on another C_m-layer. To the automorphism φ_0 there naturally corresponds a permutation π_0 of rows of M. We complete a 2-colouring of edges of $K_{1,n} \Box C_m$ such that the corresponding matrix M has different columns. However, there might exist a unique permutation π_1 of columns of M such that the composition $\pi_0 \circ \pi_1$ does not change the matrix M, thus defining a non-trivial automorphism of $K_{1,n} \Box C_m$. This is always the case when $n = 2^{2m}$. Then we certainly need a third colour, and it suffices to put it on the C_m^v-layer to obtain a distinguishing 3-colouring. If $n < 2^{2m}$, we remove a suitable number of columns from M including the column $\pi_1(c)$, i.e., an image of the column corresponding to the colouring c. This way we obtain a matrix that (together with the edge colouring of the C_m^v-layer) defines a distinguishing 2-colouring of $K_{1,n} \Box C_m$. This completes the proof. $\Box$

The same idea of the proof is used for the following more general result for $d = 2$.

Lemma 9 *Let G be a connected graph with $D'(G) = 2$. If $n \leq 2^{|G|+\|G\|}$ and the star $K_{1,n}$ is relatively prime to G, then*

$$D'(K_{1,n} \Box G) = 2.$$

Proof. Since G and $K_{1,n}$ are relatively prime, the automorphism group of the Cartesian product is generated only by automorphisms of the factors. Let u be the central vertex of the star $K_{1,n}$ and let $u_1, \ldots, u_n$ be its pendant vertices. Clearly, the G^u-layer is mapped onto itself by every $\varphi \in \mathrm{Aut}(K_{1,n} \Box G$. We start with colouring the G^u-layer with a distinguishing 2-colouring. This guarantees that all automorphisms generated by automorphisms of G are broken.

For $n = 1$ we obtain the claim by Proposition 6. Let then $n \geq 2$. Analogously to the proofs of the previous propositions, to complete a 2-colouring of $D'(K_{1,n} \Box G)$ it suffices to define a binary matrix M with $|G| + \|G\|$ rows and n columns, where the initial $\|G\|$ elements in the j-th column are colours of edges of the G^{u_j}-layer, and the other $|G|$ elements are colours of the edges of $K_{1,n}$-layers incident to consecutive vertices of the G^{u_j}-layer. As $n \leq 2^{|G|+\|G\|}$, we can choose a binary matrix M with distinct columns, thus breaking all permutations of $K_{1,n}$-layers. $\Box$

The distinguishing index of the Cartesian product of two stars can be arbitrarily large if the sizes of factors differ sufficiently, as the following observation shows.

Proposition 10 *If $m \geq 1$ and $n > d^{2m+1}$, then $D'(K_{1,m} \Box K_{1,n}) > d$.*

Proof. If $d = 1$, then the conclusion trivially follows from the fact that $|\operatorname{Aut}(K_{1,m}\Box K_{1,n})| \geq m!n! > 1$. If $d \geq 2$, then $m \neq n$ and the factors are relatively prime. Let v be the central vertex of $K_{1,n}$. Every d-colouring of the edges of $K_{1,m}\Box K_{1,n}$ except the $K_{1,m}^v$-layer can be fully described by a matrix $M = [a_{i,j}]$ of n rows and $2m+1$ columns, where for every $i = 1, \ldots, n$ the entries $a_{i,1}, \ldots, a_{i,m+1}$ are colours of edges in $K_{1,n}$-layers incident to consecutive vertices of the i-th $K_{1,m}$-layer, and $a_{i,m+2}, \ldots, a_{i,2m+1}$ are colours of consecutive edges of the i-th $K_{1,m}$-layer. As the entries of M belong to a d-element set and $n > d^{2m+1}$, each such matrix has to have at least two identical rows. Then a transposition of corresponding $K_{1,m}$-layers is an automorphism of $K_{1,m}\Box K_{1,n}$ that preserves the colouring. $\Box$

We terminate this section with two lemmas which will be useful in the next sections.

Lemma 11 *Let G and H be connected, relatively prime graphs with $D'(G) = D'(H) = 2$. Then $D'(G\Box H) = 2$.*

Proof. We colour one G-layer and one H-layer with distinguishing 2-colourings. The remaining edges can be coloured arbitrarily. Such a colouring breaks all permutations of both horizontal and vertical layers. Since G and H are relatively prime, it follows from Theorem 2 that this colouring breaks all automorphisms of $G\Box H$. $\Box$

Lemma 12 *Let G and H be two connected graphs such that G is prime, $|G| \leq \|H\| + 1$ and $D'(H) = 2$. Then $D'(G\Box H) = 2$.*

Proof. We first colour H-layers of the graph $G\Box H$. There are at least two H-layers, so we colour all edges of one layer blue, all edges of another one with a distinguishing 2-colouring. If there are more H-layers, then we colour them such that each of them has a different number of blue edges (counting the H-layers coloured previously). It is possible since $|G| \leq \|H\| + 1$. Next, we colour all edges in every G-layer red.

All automorphisms of the Cartesian product generated by the automorphisms of H are broken, since one H-layer assumes a distinguishing colouring. Also, no H-layers can be interchanged as every H-layer has different number of blue edges.

If H has a factor H' isomorphic to G, then $G\Box H$ has an automorphism interchanging H' with G. However, since all G-layers have only red edges and there exists an H-layer with only blue edges, such an automorphism does not preserve this colouring. $\Box$

3 The Cartesian powers

Let us start with the Cartesian powers of K_2. Recall that the k-dimensional hypercube is isomorphic to K_2^k and denoted by Q_k. As mentioned earlier, the distinguished index is not defined for $K_2 = Q_1$. Clearly, $D'(Q_2) = 3$ since $Q_2 = C_4$. The following result was proved in [13].

Theorem 13 *[13] If G is a traceable graph of order at least even, then $D'(G) = 2$.*

Proposition 14 *If $k \geq 3$, then $D'(Q_k) = 2$.*

Proof. For $k \geq 3$, a hypercube Q_k is hamiltonian and has at least eight vertices. Therefore, $D'(Q_k) = 2$ by Theorem 13. $\Box$

The distinguishing index of the square of cycles and for arbitrary powers of complete graphs with respect to the Cartesian, direct and strong products has been already considered by Pilśniak [12]. In particular, she proved that $D'(C_m^2) = 2$ for $m \geq 4$, and $D'(K_n^k) = 2$ for any $n \geq 4$ and $k \geq 2$. Here we consider the Cartesian powers of arbitrary connected graphs.

Lemma 15 *If G is a connected prime graph with $|G| \geq 3$, then $D'(G^k) = 2$ for every $k \geq 2$.*

Proof. The proof goes by induction on k. Let $k = 2$. There are n horizontal and n vertical layers, where $n = |G|$.

Suppose first that G contains a cycle, i.e., $\|G\| \geq n$. We colour horizontal G-layers with two colours such that each of them has a different number of blue edges between 0 and $n-1$. The other edges are coloured such that every vertical G-layer has a different number of blue edges between 1 to n. As every horizontal G-layer has a different number of blue edges they cannot be interchanged. The same is true for vertical G-layers. Therefore automorphisms of the Cartesian product generated by automorphisms of G are broken. Our colouring also breaks interchanging the factors, since there exists a completely red horizontal G-layer but no such vertical G-layer.

Assume now that G is a tree. Every tree has either a central vertex or a central edge fixed by every automorphism. In case of a tree with a central vertex v, we colour the edges of G^2 such that consecutive horizontal layers have $0, \ldots, n-1$ blue edges, and consecutive vertical layers have $0, \ldots, n-1$ blue edges in such a way that the horizontal G^v-layer and the vertical G^v-layer have all edges coloured red and blue, respectively. The vertex (v, v) is fixed by every automorphism of G^2, hence this colouring is distinguishing. If G has a central edge $e_0 = uv$, we colour the edge $(u, u)(v, u)$ red and the remaining three edges of the subgraph $e_0 \Box e_0$ blue. The vertical and horizontal G^v-layers have all edges blue and red, respectively. The remaining edges of G^2 are coloured as in the case of a tree with a central vertex. Such colouring forbids exchange of the horizonal layers with the vertical layers. Thus $D'(G^2) = 2$.

For the induction step, we apply Lemma 12 by taking $H = G^{k-1}$ since $|G| \leq \|G^{k-1}\| + 1$. $\Box$

Let us now state the main theorem of this section that solves the problem of the distinguishing index of the k-th Cartesian power of a connected graph.

Theorem 16 *Let G be a connected graph and $k \geq 2$. Then*

$$D'(G^k) = 2$$

with the only exception: $D'(K_2^2) = 3$.

Proof. Let $G = G_1^{p_1} \Box G_2^{p_2} \Box \ldots \Box G_r^{p_r}$, where $p_i \geq 1, i = 1, \ldots, r$, be the prime factor decomposition of G.

Assume first that $G_i \neq K_2$, $i = 1, 2, \ldots, r$. Then for every i we have $D'(G_i^{kp_i}) = 2$ due to Lemma 15. By repetitive application of Lemma 11 we get $D'(G^k) = 2$ since $G_i^{kp_i}$ and $G_j^{kp_j}$ are relatively prime if $i \neq j$.

Suppose now that G has a factor isomorphic to K_2, say $G_1 = K_2$. If $p_1 \geq 2$, then $D'(K_2^{kp_1}) = 2$ and again $D'(G^k) = 2$ by Lemma 11 applied to $K_2^{kp_1}$ and $G_2^{p_2} \Box \ldots \Box G_r^{p_r}$. The same argument applies in case $p_1 = 1$ and $k \geq 3$. Finally, if $p_1 = 1$ and $k = 2$ we apply Lemma 12 twice and we also get $D'(G^k) = 2$ unless $r = 1$, i.e., $G^k = K_2^2$. $\Box$

4 The Cartesian product

In this section we investigate sufficient conditions on the sizes of factors of the Cartesian product of two graphs to have the distinguishing index equal to two. We begin with a result for trees. Observe first that, by Theorem 2, the Cartesian product of two non-isomorphic asymmetric trees is an asymmetric graph, so its distinguishing index is equal to 1.

Theorem 17 *Let T_m and T_n be trees of size m and n, respectively. If*

$$2 \leq m \leq n \leq 2^{2m+1} - \left\lceil \frac{m}{2} \right\rceil + 1,$$

then $D'(T_m \Box T_n) \leq 2$.

Proof. If T_m is isomorphic to T_n, then the conclusion follows from Lemma15. Therefore, assume that T_m and T_n are non-isomorphic. Then they are relatively prime, and it is enough to prove the existence of a 2-colouring of edges of $T_m \Box T_n$ that breaks the automorphisms generated by automorphisms of T_m and those generated by automorphisms of T_n.

In the proof we use the following well-known fact. In a rooted tree, if a parent vertex is fixed by every automorphism preserving a given colouring and its children cannot be permuted, then the children are also fixed.

Assume first that $n = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$. We choose a root u_0 of T_m as follows. If T_m has a central vertex, then we take it as a root u_0. If T_m has a central edge, then we choose one of its end-vertices as u_0 and the other one as u_1. Then we choose an enumeration $u_0, \ldots, u_m$ of vertices of the rooted tree T_m satisfying the following condition: if u_i is a parent of u_j, then $i < j$. We enumerate the edge $u_i u_j = e_j$. Thus $E(T_m) = \{e_1, \ldots, e_m\}$. Let v_0 be a root of T_n chosen by the same rule as the root u_0 of T_m. Then we analogously enumerate vertices and edges of T_n to obtain $V(T_n) = \{v_0, \ldots, v_n\}$, $E(T_n) = \{\varepsilon_1, \ldots, \varepsilon_n\}$.

We begin with colouring of the $T_m^{v_0}$-layer by putting colour 0 on the edges $e_i, i = 1, \ldots, \lceil \frac{m}{2} \rceil$, and colour 1 on the remaining edges of this layer. It is easy to see that we can choose such an enumeration of vertices, and hence of edges, that the root u_0 is fixed by every automorphism of T_m preserving this colouring. Indeed, this is obvious if u_0 is a central vertex; if $e_1 = u_0 u_1$ is a central edge of T_m, then we enumerate the vertices such that $u_1, \ldots, u_{\lfloor \frac{m}{2} \rfloor}$ belong to the same subtree of $T_m - e_1$, therefore our colouring breaks all automorphisms of T_m reversing the end-vertices of e_1.

Then, we similarly colour the $T_n^{u_0}$-layer with 0 and 1 in such a way that the vertex (u_0, v_0) is fixed by every automorphism of $T_m \square T_n$ preserving this partial colouring. Hence, the $T_m^{v_0}$-layer, as well as the $T_n^{u_0}$-layer, is mapped onto itself by every $\varphi \in \mathrm{Aut}(T_m \square T_n)$ preserving this colouring.

Next, we colour the other layers. Consider the set S of all 2^{2m+1} binary sequences of length $2m+1$. Each $T_m^{v_i}$-layer with $i \geq 1$ is assigned a distinct sequence

$$s_i = (a_0, a_1, \ldots, a_m, b_1 \ldots, b_m)$$

from S, where $a_j, j = 0, \ldots, m$, is the colour of the edge ε_i joining the vertex (u_j, v_i) with its parent in the $T_n^{u_j}$-layer (observe that a_0 has been already defined for all $i \geq 1$), and $b_j, j = 1, \ldots, m$ is the colour of the edge of the $T_m^{v_i}$-layer corresponding to e_j. To break all permutations of T_n-layers we delete some sequences from the set S. First observe that the sum of each coordinate taken over all sequences in S is the same (and equal to 2^{2m}). Clearly, a $T_n^{u_j}$-layer and a $T_n^{u_{j'}}$-layer cannot be permuted whenever $j \leq \lceil \frac{m}{2} \rceil < j'$ since the edges e_j and $e_{j'}$ in the $T_m^{v_0}$-layer have different colours.

Consider the set $A = \{s^k \in S \ : \ k = 1, \ldots, \lceil \frac{m}{2} \rceil - 1\}$, where $s^k = (a_0, a_1, \ldots, a_m, b_1, \ldots, b_m)$ is a sequence such that

$$a_j = a_{\lceil \frac{m}{2} \rceil + j} = 1, \quad j = 1, \ldots, k,$$

and all other elements of s^k are equal to 0. Thus $|S \setminus A| = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$. We use the set $S \setminus A$ to colour $T_m^{v_i}$-layers, $i = 1, \ldots, 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$, hence the numbers of edges coloured with 1 is distinct for every pair of T_n-layers that could be permuted. Thus, all edges in $T_m \square T_n$ are coloured, and we obtain a distinguishing 2-colouring of $T_m \square T_n$, when $n = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$.

Now, assume that the difference $l = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1 - n$ is positive. We have to choose l sequences from $S \setminus A$ that will not be used in the colouring. To do this denote $\overline{0} = 1, \overline{1} = 0$. A pair of sequences

$$(a_0, a_1, \ldots, a_m, b_1, \ldots, b_m), \quad (a_0, \overline{a_1}, \ldots, \overline{a_m}, b_1, \ldots, b_m)$$

from $S \setminus A$ is called *complementary*. When l is even, we choose $\frac{l}{2}$ complementary pairs. When l is odd, we choose the sequence $(0, \ldots, 0) \in S \setminus A$ and $\frac{l-1}{2}$ complementary pairs. It is easily seen that all permutations of layers in $T_m \square T_n$ are broken by this colouring. $\square$

The bound $2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$ for the size of a larger tree is perhaps not sharp. However, it cannot be improved much since it follows from Proposition 10 that $D'(K_{1,m} \square K_{1,n}) \geq 3$ whenever $n > 2^{2m+1}$.

We now consider the Cartesian product of connected graphs in general. We first formulate a result for relatively prime factors.

Lemma 18 *Let G and H be connected, relatively prime graphs such that*

$$3 \leq |G| \leq |H| \leq 2^{|G|}\big(2^{\|G\|} - 1\big) - |G| + 2.$$

Then $D'(G \square H) \leq 2$.

Proof. Denote $V(G) = \{u_1, \ldots, u_{|G|}\}$, $E(G) = \{e_1, \ldots, e_{\|G\|}\}$, $V(H) = \{v_1, \ldots, v_{|H|}\}$, $E(H) = \{\varepsilon_1, \ldots, \varepsilon_{\|H\|}\}$. Assume that v_1 is a root of a spanning tree T_H of the graph H, and the vertices of H are enumerated according to the rooted tree T_H, i.e., each child has the index greater than that of its parent. As G and H are relatively prime, the only automorphisms of $G \square H$ are some permutations of G-layers and H-layers.

We first colour the edges of the G^{v_1}-layer with a sequence

$$(b_1, \ldots, b_{\|G\|}) = (1, \ldots, 1).$$

We shall not use this sequence to colour the edges of any other G-layer any more. Thus the G^{v_1}-layer will be mapped onto itself by every automorphism of $G \square H$ preserving the colouring.

From now on, we proceed similarly as in the proof of Theorem 17. For $i = 2, \ldots, n$, the G^{v_i}-layer will be assigned a distinct sequence of colours

$$(a_1, \ldots, a_{|G|}, b_1, \ldots, b_{\|G\|}),$$

where a_j is a colour of the edge joining the vertex (u_j, v_i) to its parent in the rooted tree T_H in the H^{u_j}-layer, and b_j is a colour of e_j in the G^{v_i}-layer. We have $2^{|G|}\big(2^{\|G\|} - 1\big)$ such sequences, as we excluded all sequences of the form $(a_1, \ldots, a_{|G|}, 1, \ldots, 1)$. Thus all permutations of G-layers are broken. To break permutations of H-layers, we also exclude all sequences $s^k = (a_1, \ldots, a_{|G|}, b_1, \ldots, b_{\|G\|})$ with $a_1 = \ldots = a_k = 1$ and $a_{k+1} = \ldots = a_{|G|} = b_1 = \ldots = b_{\|G\|} = 0$, for every $k = 1, \ldots, |G| - 1$. We have $2^{|G|}\big(2^{\|G\|} - 1\big) - (|G| - 1)$ sequences to colour $|H| - 1$ G-layers. Depending on the size of $|H|$, we also exclude a suitable number of complementary pairs of sequences

$$(a_1, \ldots, a_{|G|}, b_1, \ldots, b_{\|G\|}), \quad (\overline{a_1}, \ldots, \overline{a_{|G|}}, b_1, \ldots, b_{\|G\|})$$

and, possibly, a sequence $(0,\ldots,0)$. Thus we obtain a colouring of $G\Box H$ with a set of sequences such that the number of 1's is distinct in any of the initial $|G|$ coordinates. Therefore, no permutation of H-layers preserves this colouring. Hence, this is a distinguishing 2-colouring of $G\Box H$. $\Box$

Finally, we state the main result of this section.

Theorem 19 *Let G and H be connected graphs such that*

$$2 \leq |G| \leq |H| \leq 2^{|G|}\big(2^{\|G\|}-1\big) - |G| + 2.$$

Then $D'(G\Box H) \leq 2$ unless $G = H = K_2$.

Proof. If $G = K_2$, then $|H| \leq 4$. If $H \neq K_4$, then either $D'(H) = 2$ or H is a cycle or a star, and these cases were already settled in Section 2. To construct a distinguishing 2-colouring of $K_2\Box K_4$, colour one edge in one K_4-layer and two adjacent edges in the other K_4-layer red, and all remaining edges blue.

Let then $|G| \geq 3$. The case when G and H are relatively prime was settled in Lemma 18. Therefore, we focus here on the situation when G and H have at least one common factor. Then $D'(G\Box H) \geq 2$ since the automorphism group of $G\Box H$ is non-trivial. Let $G = G_1^{k_1}\Box\ldots\Box G_t^{k_t}$ and $H = H_1^{l_1}\Box\ldots\Box H_s^{l_s}$ be the prime factor decompositions of G and H, respectively. Assume that the initial r factors are common, i.e., $G_i = H_i$ for $i = 1,\ldots,r$. Denote

$$G_{II} = G_1^{k_1}\Box\ldots\Box G_r^{k_r}, \qquad H_{II} = H_1^{l_1}\Box\ldots\Box H_r^{l_r}.$$

Thus $G = G_I\Box G_{II}$ and $H = H_I\Box H_{II}$. We use the following notation

$$n_1 = |G_I|, \quad n_2 = |G_{II}|, \quad m_1 = |H_I|, \quad m_2 = |H_{II}|.$$

We first show that the distinguishing index of the Cartesian product

$$G_{II}\Box H_{II} = G_1^{l_i+k_1}\Box\ldots\Box G_r^{l_r+k_r}$$

is equal to 2. If $G_{II}\Box H_{II} = K_2^2$, then $|H_I| \geq 2$ and the graphs $G_I\Box K_2^2$ and H_I satisfy the assumptions of Theorem 18, hence $D'(G\Box H) = 2$, unless $|G_I\Box K_2^2| > |H_I|$, that is $|H_I| < 4|G_I|$. In the latter case, we can also apply Theorem 18 for the graphs G_I and H_I which are relatively prime and satisfy the inequalities $|G_I| \leq |H_I| \leq 2^{|G_I|}(2^{\|G_I\|}-1) - |G_I| + 2$ unless $|G_I| = 2$

and $\leq |H_I| \leq 7$, i.e., $G\Box H = K_2^3\Box H_I'$, where H_I' is prime, so we can apply Proposition 14 and Lemma 12. In any case $D'(G\Box H) = 2$.

If $G_i^{l_i+k_i} \neq K_2^2$ for every $i = 1,\dots,r$, then $D'(G_i^{l_1+k_i}) = 2$ due to Theorem 16, and hence $D'(G_{II}\Box H_{II}) = 2$ by repeated application of Lemma 11. If $G_1^{l_1+k_1} = K_2^2$, then analogously $D'(G_2^{l_2+k_2}\Box\dots\Box G_r^{l_r+k_r}) = 2$, hence $D'(G_{II}\Box H_{II}) = 2$ by applying Proposition 6 twice.

Consider now the graphs $G' = G_I\Box G_{II}\Box H_{II}$ and $H' = H_I$. Clearly, they are relatively prime and

$$|H'| < |H| \leq 2^{|G|}\big(2^{\|G\|} - 1\big) - |G| + 2 < 2^{|G'|}\big(2^{\|G'\|} - 1\big) - |G'| + 2.$$

If also $|G'| = n_1n_2m_2 \leq m_1 = |H'|$, then graphs G' and H' satisfy the conditions of Lemma 18, and consequently, $D'(G\Box H) = D'(G'\Box H') = 2$. Then suppose that $n_1n_2m_2 > m_1$. We consider two cases here.

Assume first that $n_1 \leq n_2m_2$, i.e., $|G_I| \leq |G_{II}\Box H_{II}|$. Hence, $|G_I| \leq \|G_{II}\Box H_{II}\|+1$, and by repeated application of Lemma 12 we get $D'(G') = 2$. As $|H'| < |G'|$, we infer again from Lemma 12 that $D'(G\Box H) = D'(G'\Box H') = 2$.

In the second case, i.e., when $n_2m_2 < n_1$, suppose first that

$$m_1 = |H_I| \leq 2^{|G_I|}\big(2^{\|G_I\|} - 1\big) - |G_I| + 2.$$

Then $D'(G_I\Box H_I) \leq 2$ since the assumptions of Lemma 18 is satisfied whenever $|G_I| \leq |H_I|$. Recall that also $D'(G_{II}\Box H_{II}) = 2$ and graphs $G_I\Box H_I$ and $G_{II}\Box H_{II}$ are relatively prime. Hence $D'(G\Box H) = 2$ by Lemma 11. Otherwise, if $m_1 > 2^{|G_I|}\big(2^{\|G_I\|} - 1\big) - |G_I| + 2$, then we arrive at the sequence of inequalities

$$m_1 < n_1n_2m_2 \leq n_1^2 < 2^{n_1}(2^{n_1} - 1) - n_1 + 2 \leq 2^{|G_I|}\big(2^{\|G_I\|} - 1\big) - |G_I| + 2 < m_1,$$

which gives a contradiction.

Then suppose that $|G_I| = n_1 > m_1 = |H_I|$ (and still $n_2m_2 < n_1$). Let $G'' = G_I$ and $H'' = G_{II}\Box H_I\Box H_{II}$. Clearly, $|G''| \leq |H''|$ since $|G| \leq |H|$. Moreover, we have

$$|H''| = n_2m_2m_1 < n_1m_1 < n_1^2 < 2^{|G''|}\big(2^{\|G''\|} - 1\big) - |G''| + 2.$$

It follows from Lemma 18 that $D'(G\Box H) = D'(G''\Box H'') = 2$. $\Box$

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