MATEMATYKA
DYSKRETNA
www.ii.uj.edu.pl/preMD/

Olivier BAUDON, Julien BENSMAIL
Jakub PRZYBYŁO and Mariusz WOŹNIAK

On decomposing regular graphs into locally irregular subgraphs

Preprint Nr MD 065
(otrzymany dnia 10.03.2013)

Kraków
2013

On decomposing regular graphs into locally irregular subgraphs

Olivier Baudon, Julien Bensmail
Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France
CNRS, LaBRI, UMR 5800, F-33400 Talence, France
Jakub Przybyło*, Mariusz Woźniak†
AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

March 10, 2013

Abstract

A *locally irregular graph* is a graph whose adjacent vertices have distinct degrees. We say that a graph G *can be decomposed into* k *locally irregular subgraphs* if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph in G. We characterize all connected graphs which cannot be decomposed into locally irregular subgraphs. These are all of odd size and include paths, cycles and a special class of graphs of maximum degree 3. Moreover we conjecture that apart from these exceptions all other connected graphs can be decomposed into 3 locally irregular subgraphs. Using a combination of a probabilistic approach and some known theorems on degree constrained subgraphs of a given graph, we prove this statement to hold for all regular graphs of degree at least 10^7. We also support this conjecture by showing that decompositions into three or two such subgraphs might be indicated e.g. for some bipartite graphs (including trees), complete graphs and cartesian products of graphs with this property (hypercubes for instance). We also investigate a total version of this problem, where in some sense also the vertices are being prescribed to particular subgraphs of a decomposition. The both concepts are closely related to the known 1-2-3 Conjecture and 1-2 Conjecture, respectively, and other similar problems concerning edge colourings. In particular, we improve the result of Addario-Berry, Aldred, Dalal and Reed [J. Combin. Theory Ser. B 94 (2005) 237-244] in the case of regular graphs.

Keywords: locally irregular graph, graph decomposition, edge set partitioning, 1-2-3 Conjecture, 1-2 Conjecture

*Corresponding author. E-mail: przybylo@wms.mat.agh.edu.pl.

†The research of the third and fourth authors were partially supported by the Polish Ministry of Science and Higher Education.

1 Introduction

All graphs considered are simple and finite. We follow [10] for the notations and terminology not defined here. Consider a graph $G = (V, E)$. It is well known that if its order n is at least two, then it cannot be (completely) *irregular*, i.e., it must contain a pair of vertices of the same degree. By a *locally irregular graph* we shall mean a graph such that the degree of every vertex is distinct from the degrees of all of its neighbours. In other words, it is a graph in which the adjacent vertices have distinct degrees. Such graphs exist for every order n. A natural antonym of the class of these is the family of regular graphs. In this paper we investigate decompositions of regular, or more generally any graphs into locally irregular subgraphs. More precisely, we say that G can be *decomposed into k locally irregular subgraphs* if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph, i.e., $E = E_1 \cup E_2 \cup \ldots \cup E_k$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $H_i := (V, E_i)$ is locally irregular for $i = 1, 2, \ldots, k$. Note that instead of decomposing the graph G, we may alterably paint its edges with k colours so that every colour class induces a locally irregular subgraph in G. Such colouring shall be called a *locally irregular k-edge colouring* of G. The colour classes of this naturally define locally irregular subgraphs of G making up its decomposition. Thus the two notions shall be used equivalently. Note that if an edge $uv \in E$ has colour i assigned by a locally irregular edge colouring, then the numbers of edges coloured with i incident with u and v must be distinct. As usual we shall be most interested in the least number of colours necessary to create such a colouring. However, not every graph admits any such colour assignment, this does not exist e.g. for the path P_2 (on 2 vertices). Other exceptions are discussed further on. Apart from these, we suspect that three colours ($k = 3$) are sufficient for all remaining graphs, cf. Conjecture 3.4. Intriguingly, the subject of our investigations binds several other related problems, which in fact motivated our research.

1.1 1-2-3 Conjecture

Consider another concept of introducing local irregularity in a graph by means of edge colourings (or *weightings*). Let $c : E \to \{1, 2, \ldots, k\}$ be an edge colouring of G with positive integers. For every vertex v we then denote by $s_c(v) := \sum_{u \in N(v)} c(uv)$ the sum of its incident colours and call it the *weighted degree* of v. We say that c is a *neighbour sum distinguishing k-edge colouring* of G if $s_c(u) \neq s_c(v)$ for all adjacent vertices u, v in G. Another interpretation of this concept, introduced by Karoński, Łuczak and Thomason [17], asserts that instead of assigning an integer to every edge, we multiply it the corresponding number of times in order to create a *locally irregular multigraph* of G, i.e., a multigraph whose neighbours have distinct degrees. This problem came to life as a descendant of the graph invariant known as the *irregularity strength* of a graph, where as above, given a graph, one strives to create of it a multigraph in which *all* vertices have distinct degrees, see e.g. [4, 9, 11, 15, 18, 19, 21] for further details and some of the most up-to-date results and open problems concerning this parameter. It is also worth mentioning that the concept of the irregularity strength was motivated by the study of Chartrand, Erdős, Oellermann et al. concerning 'irregular graphs' (see [5, 6, 8]), whose research are also closely related to ours.

In [17] Karoński, Łuczak and Thomason posed the following elegant problem, known as the *1-2-3 Conjecture*.

Conjecture 1.1 (1-2-3 Conjecture) *There exists a neighbour sum distinguishing* 3*-edge colouring of every graph* G *containing no isolated edges.*

Thus far it is known that a neighbour sum distinguishing 5-edge colouring exists for every graph without isolated edges, see [16]. On the other hand, Addario-Berry, Dalal and Reed proved in [3] the following result for random graphs.

Theorem 1.2 *If* G *is a random graph (chosen from* $G_{n,p}$ *for a constant* $p \in (0,1)$*), then asymptotically almost surely, there exists a neighbour sum distinguishing* 2*-edge colouring of* G*.*

This fact gets even more interesting in view of our research if combined with the following straightforward observation.

Observation 1.3 *If* G *is a* ***regular*** *graph, then there exists a neighbour sum distinguishing* 2*-edge colouring of* G *if and only if there exists a locally irregular* 2*-edge colouring of* G*.*

To see that it holds, consider an edge colouring $c : E \to \{1,2\}$ of a regular graph G and any edge $uv \in E$. If c is neighbour sum distinguishing, then $s_c(u) \neq s_c(v)$ implies that u, v must be incident with different numbers of edges coloured with 1 or 2, but since $d(u) = d(v)$, they must differ in frequencies of the both colours. Thus u, v have distinct degrees in the subgraph induced in G by the class of the colour $c(uv)$, regardless of whether $c(uv) = 1$ or $c(uv) = 2$. The colouring c must therefore also be locally irregular. A similar argument works the other way round. Note that the corresponding equivalence does not hold in case of k-edge colourings if $k \geq 3$ (consider e.g. C_5).

There were quite a few attempts to tackle 1-2-3 Conjecture before the set of required colours was narrowed down to $\{1,2,3,4,5\}$, see [1, 3, 16, 17, 24]. In fact in the first paper on 1-2-3 Conjecture [17] the authors proved only that a finite collection of (183 independent over the field of the rationals) real numbers admitted as edge colours always suffice to distinguish neighbours by sums. For this goal, they showed that for every graph without isolated edges, there is a colouring $c : E \to \{1, 2, \ldots, k\}$, with $k = 183$, such that the end-vertices of every edge obtain distinct *multisets* of their incident colours. Such colouring c shall be called a *neighbour multiset distinguishing k-edge colouring.* The mentioned result was then greatly improved by Addario-Berry et al. [2], who proved the following.

Theorem 1.4 *There exists a neighbour multiset distinguishing* 4*-edge colouring of every graph G containing no isolated edges.*

Theorem 1.5 *There exists a neighbour multiset distinguishing* 3*-edge colouring of every graph G of minimum degree $\delta \geq 1000$.*

Note that every locally irregular k-edge colouring is also a neighbour multiset distinguishing k-edge colouring. This however does not have to be the case the other way round, unless we narrow down our concern to 2-edge colourings of regular graphs. Though the problem of locally irregular decompositions is interesting for general graphs, in this paper we focus mainly, but not exclusively on regular ones, which are in some sense the least irregular among all. The main result of this paper is a strengthening of Theorem 1.5 for this family of graphs. Namely, we prove that if G is a regular graph of sufficiently large minimum degree, then there exists its locally irregular 3-edge colouring (which is the more a neighbour multiset distinguishing 3-edge colouring of G), see Theorem 5.1 in section 5. Its proof combines probabilistic approach with some known theorems on degree constrained subgraphs of a given graph from [1], see section 4. We also settle for which graphs a decomposition to any number of locally irregular subgraphs exists at all. This and other results on locally irregular decompositions, concerning mainly bipartite graphs, complete graphs and cartesian products of graphs are contained in section 3. We begin however with a short discussion on a total version of our problem, since it was in fact the starting point for introducing all concepts contained in this paper. In the following, given two graphs $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$, usually subgraphs of a host graph G, by $H_1 \cup H_2$ we shall mean the graph $(V_1 \cup V_2, E_1 \cup E_2)$. Moreover, we shall write $H_2 \subset H_1$ if $V_2 \subset V_1$ and $E_2 \subset E_1$, and in case of $H_2 \subset H_1$, we shall also write $H_1 - E(H_2)$ to denote the graph obtained from H_1 by removing the edges of H_2. From now on, given a subset E' of edges of a given graph $G = (V, E)$, the *graph induced by E'* shall be understood as $G' := (V', E')$ where V' is the set of the end-vertices of all edges in E' (note that the 'local irregularity' of G' is not altered by whether its vertex set is V or V').

2 1-2 Conjecture and Locally Irregular Total Colourings

In [23] the following problem related to 1-2-3 Conjecture was introduced. Let $c : E \cup V \to \{1, 2, \ldots, k\}$ be a total colouring of a graph $G = (V, E)$

with positive integers. For every vertex v we then denote by $t_c(v) := c(v) + \sum_{u \in N(v)} c(uv)$ the sum of its incident colours and the colour of v, and call it the *total weighted degree* of v. We say that c is a *neighbour sum distinguishing k-total colouring* of G if $t_c(u) \neq t_c(v)$ for all adjacent vertices u, v in G.

Conjecture 2.1 (1-2 Conjecture, [23]) *There exists a neighbour sum distinguishing 2-total colouring of every graph G.*

In this context, the following best upper bound is due to Kalkowski [14].

Theorem 2.2 *There exists a neighbour sum distinguishing 3-total colouring of every graph G.*

For any total colouring $c : E \cup V \to \{1, 2, \ldots, k\}$, $v \in V$ and $i \in \{1, 2, \ldots, k\}$, by $c_i(v)$ we shall mean the number of the elements of the set $\{v\} \cup \{vw : w \in N(v)\}$ coloured with i (hence $c_1(v) + c_2(v) + \ldots + c_k(v) = d(v) + 1$). The total colouring c shall be called a *locally irregular k-total colouring* when, similarly as in the case of edge colourings, for every edge $uv \in E$, if $c(uv) = i$, then $c_i(u) \neq c_i(v)$. We ask whether the following is true.

Conjecture 2.3 *There exists a locally irregular 2-total colouring of every graph G.*

The main reason for our interest with locally irregular total colourings, and in particular with the conjecture above, were our endeavours towards proving 1-2 Conjecture for **regular** graphs (see [22, 23]), for which Conjectures 2.1 and 2.3 are equivalent.

Observation 2.4 *If G is a **regular** graph, then there exists a neighbour sum distinguishing 2-total colouring of G if and only if there exists a locally irregular 2-total colouring of G.*

This holds by the same reasoning as Observation 1.3. In [23] it was in particular proven that 1-2 Conjecture is true for complete graphs, 3-colourable graphs (i.e. with $\chi(G) \leq 3$) and 4-regular graphs. Thus by observation 2.4 we obtain the following conclusion.

Corollary 2.5 *If G is a d-regular graph with $d \leq 4$ or a complete graph, then there exists its locally irregular 2-total colouring.*

Proposition 2.6 *If $G = (A, B; E)$ is a bipartite graph, then there exists its locally irregular 2-total colouring.*

Proof. We define the total colouring c of G as follows. First let us colour all edges of G with colour 1. Then we colour with 1 every vertex v such that $v \in A$ and $d(v)$ is odd or $v \in B$ and $d(v)$ is even. The remaining vertices of G are coloured by 2. Note that this way every edge of $uv \in E$ with $u \in A$, $v \in B$ is coloured with 1, while $c_1(u) \equiv 0 \pmod 2$ and $c_1(v) \equiv 1 \pmod 2$, hence $c_1(u) \neq c_1(v)$. ■

Observation 2.7 *If there exists a locally irregular k-edge colouring of a graph G, then there exists a locally irregular k-total colouring of G.*

Indeed, given any locally irregular edge colouring of G it is sufficient to extend it by colouring every vertex with the same colour, say 1. By Observation 2.7 all our further results concerning edge colourings (decompositions) transfer directly to the total ones.

Let us finally note that we may also use the terminology of *decompositions* in case of the concept investigated in this section if we first introduce the following definitions. A *total graph* is an ordered triple (V_0, V_1, E), where V_0, V_1 are called the sets of *empty* and *solid vertices*, resp., $V_0 \cap V_1 = \emptyset$, and E, the set of *edges*, is a subset of $\binom{V_0 \cup V_1}{2}$ $(E \cap (V_0 \cup V_1) = \emptyset)$. Such total graph is called *locally irregular* if for every edge $uv \in E$, the degree of u is distinct from the degree of v, where by the *degree* of a vertex w we mean the number of edges in E containing w plus 1 if $w \in V_1$ (or plus 0 if $w \in V_0$). Then every locally irregular k-total colouring $c : E \cup V \to \{1, 2, \ldots, k\}$ of G is equivalent to a decomposition of G into k locally irregular total (sub-)graphs $H_1, H_2, \ldots, H_k$, where $H_i = (V_0^i, V_1^i, E^i)$ with $E^i = \{e \in E : c(e) = i\}$, $V_1^i = \{v \in V : c(v) = i\}$ and $V_0^i = V \setminus V_1^i$. We thus believe that virtually every graph G can be decomposed into two locally irregular total graphs, see Conjecture 2.3.

3 Decompositions into Locally Irregular Subgraphs

As mentioned, there are some exceptions in case of edge colourings, i.e., there exist graphs which cannot be decomposed into any number of locally irregular subgraphs.

Proposition 3.1 *If P_n is a path with n vertices, then there exists its locally irregular 2-edge colouring if n is odd, but there does not exist any locally irregular edge colouring of P_n otherwise.*

Proof. Note that all connected subgraphs of any path are paths too, while the only locally irregular path is P_3 (and P_1). Thus every locally irregular subgraph of a path must contain an even number of edges, hence the second part of the thesis follows. If n is odd, hence P_n is of even length, then we colour every second pair of consecutive edges with colour 1 and the remaining every second pair of edges with colour 2. (Obviously, for P_3 and P_1 we do not need two colours.) ∎

Proposition 3.2 *If C_n is a cycle with n vertices, then there exists its locally irregular 2-edge colouring if $n \equiv 0 \pmod 4$, or a locally irregular 3-edge colouring if $n \equiv 2 \pmod 4$, but there does not exist any locally irregular edge colouring of C_n otherwise.*

Proof. If $n \equiv 0 \pmod 4$, then similarly as for paths, we colour the subsequent pairs of consecutive edges with 1 and 2 alternately. If $n \equiv 2 \pmod 4$ we do the same, but we start by painting one pair of consecutive edges with the colour 3. Since every proper subgraph of C_n is a path, the last statement from the thesis follows by the same argument as for paths. ■

All the remaining exceptions belong to the family $\mathfrak{T}$ defined inductively as follows.

- The triangle K_3 belongs to $\mathfrak{T}$.
- Every other graph of this family may be constructed by taking an auxiliary graph F which might either be a path of even length or a path of odd length with a triangle glued to one of its ends, then choosing a graph $G \in \mathfrak{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally identifying v with a vertex of degree 1 of F.

Note that every member of this family has a 'tree-like structure' and is of *odd* size. Less formally, $\mathfrak{T}$ might be characterized as the family of connected graphs whose every member G has maximum degree $\Delta \leq 3$ and *circumference* (the maximum length of its cycle) equal to 3, and which consists of a system of paths and triangles arranged so that the length of every path joining two triangles is odd, the length of every path joining a triangle with a vertex of degree 1 is even and every vertex of degree 3 belongs to exactly one triangle.

We shall now prove that no member of this family can be decomposed into locally irregular subgraphs. The proof that the remaining connected graphs (except for odd paths and cycles) can be decomposed into such subgraphs is a bit more complex and shall be included at the end of this section, see Theorem 3.13.

Observation 3.3 *If $G \in \mathfrak{T}$, then there does not exist any locally irregular edge colouring of G.*

Proof. Suppose that $G \in \mathfrak{T}$ is a minimal counterexample to the thesis, i.e., there exists a locally irregular edge colouring $c : E \to \{1, 2, \ldots, k\}$ of G, but there does not exist such colouring for any other member of $\mathfrak{T}$ with fewer edges than G. Let vertices u, v, w induce a triangle in G. Note that all edges of this triangle cannot have the same colour assigned by c, since then all of them would be neighbours of degree 2 or 3 in the subgraph induced by this colour in G, a contradiction. Thus one of these edges must have a unique colour prescribed. Without the loss of generality we may assume that $c(uv) = 1$, $c(uw) \neq 1$, $c(vw) \neq 1$ and at least one of the ends of uv, say u, is incident with one edge outside the triangle, say $uu' \in E$ with $u' \notin \{v, w\}$, which is coloured with 1 (since otherwise the subgraph induced by the colour

1 in G would contain an isolated edge). Denote by $G_{u'}$ the component of $G-u$ which contains u', and note that the subgraph G' induced in G by $V(G_{u'}) \cup \{u, v\}$ belongs to $\mathfrak{T}$ or is an odd length path. Moreover (since even if v is incident with an edge outside the triangle, then this edge cannot be coloured with 1) the decomposition into locally irregular subgraphs induced by c in G is also valid if narrowed down to its subgraph G', a contradiction. ■

All our further results seem to support the following presumption, which we believe to be true.

Conjecture 3.4 *Every connected graph G which does not belong to $\mathfrak{T}$ and is not an odd length path nor an odd length cycle can be decomposed into 3 locally irregular subgraphs.*

Theorem 3.5 *There exists a locally irregular 3-edge colouring of every complete graph K_n of order $n \geq 4$.*

Proof. The proof is inductive with respect to n. For this reason we will have to prove a slightly stronger statement. Namely, for every $n \geq 4$ we shall prove that there exists a locally irregular 3-edge colouring of K_n in which there exists no vertex whose all $(n-1)$ incident edges are coloured with 1 or there exists no vertex whose all $(n-1)$ incident edges are coloured with 2.

For K_4 we may easily find a required edge colouring with every of the colours $1, 2, 3$ used to paint two incident edges. For $n \geq 5$, we first fix a colouring guaranteed by the induction hypothesis for a subgraph of K_n isomorphic with K_{n-1}. If no vertex of K_{n-1} is incident exclusively with edges coloured with 1 (or 2), then we colour the remaining edges with 1 (or 2, resp.). It is straightforward to notice that such edge colouring of K_n fulfills our requirements. ■

Note that in the colourings of K_n, $n \geq 4$, generated by the inductive procedure above colour 3 appears only on two edges. By essentially the same reasoning we may also prove that if we remove two or even one edge from K_n, then two colours are sufficient.

Proposition 3.6 *If K_n is a complete graph of order $n \geq 4$, then there exist locally irregular 2-edge colourings of $K_n - e$ and $K_n - \{e, e'\}$, where e, e' are arbitrary edges of K_n.*

Obviously, sometimes even one colour is sufficient. The only graphs for which there exists a locally irregular 1-edge colouring are those which are locally irregular themselves. There are numerous examples of infinite classes of such graphs, e.g., the family of stars $K_{1,m}$ with $m \geq 2$. More generally, the following is true.

Proposition 3.7 *If $K_{p,q}$ is a complete bipartite graph with $q \geq 2$ (or $p \geq 2$), then there exists its locally irregular 2-edge colouring, or even a locally irregular 1-edge colouring if $p \neq q$.*

Proof. If $p \neq q$, then $K_{p,q}$ is obviously locally irregular. On the opposite, if $p = q$, then it gets locally irregular if we remove one vertex (together with its incident edges) of it. Since the removed edges form a star $K_{1,q}$ with $q \geq 2$, the thesis follows. ■

Though the class of bipartite graphs appeared to be fairly easy while investigating total colourings, see Proposition 2.6, no corresponding general result for edge version is known. By Proposition 3.1, we even know that no locally irregular edge colouring exists for infinitely many bipartite graphs, even if we narrow down our concern to trees exclusively. In [13] Havet et al. proved that there exists a neighbour multiset distinguishing 2-edge colouring for every bipartite graph with minimum degree at least three. We thus obtain the following conclusion.

Corollary 3.8 *There exists a locally irregular 2-edge colouring of every regular bipartite graph G with minimum degree $\delta \geq 3$.*

Theorem 3.9 *If T is a tree which is not an odd length path, then there exists its locally irregular 3-edge colouring.*

Proof. Let us first note that except for the case of P_2 and P_4, there exists a locally irregular 2-edge colouring for every *spidey*, i.e., a tree of radius at most two consisting of a *central vertex* of arbitrary degree, say w, and the remaining vertices of degree at most 2 which are at distance at most 2 from w. Indeed, if $d(w) \geq 3$, then the spidey is locally irregular itself, while otherwise it is just a path of length at most 4.

We now prove the theorem for all trees by induction with respect to their order n. For $n \leq 5$, every tree is a spidey, hence the thesis follows by our observation above. Assume then that $n \geq 6$ and T is not a spidey nor an odd length path.

Fix any vertex r as a *root* of T. As usual the neighbour u of a vertex v which is closer to r than v is called the *father* of v, while v is then called the *son* of u. Moreover, every vertex $v' \neq v$ with v on the unique path joining v' with r is called a *descendant* of v. Let A be the set of vertices in T which are not leaves but have no descendant at distance more than 2. For every $v \in A$, let T_v be a subtree (of radius at most 2) induced in T by v and all its descendants.

Suppose that at least one of such subtrees T_v is a spidey (with the central vertex v) other than P_2 and P_4 (hence $v \neq r$, since T is not a spidey itself). Choose any such T_v and denote the father of v in T by u. Then by the induction hypothesis we may find a locally irregular 3-edge colouring of a subtree T' induced in T by all vertices except the descendants of v unless T' is an odd length path, or a locally irregular 2-edge colouring of $T' - v$ otherwise. In both cases, a subtree induced by the remaining edges of T is a spidey different from P_2 and P_4 (in the second case it follows from the fact that T is not an odd length path) incident with at most one already coloured edge. By our observation from the first paragraph of the proof we then may use the other two colours to construct a locally irregular 2-edge colouring of the remaining part of T, hence completing the locally irregular 3-edge colouring of T.

We thus may assume that every subtree T_v, $v \in A$, is not a spidey or is isomorphic to P_2 or P_4. Note that every T_v of radius 1 is obviously a spidey, hence it must be isomorphic to P_2, i.e., $d(v) = 2$. This however means that every T_v, $v \in A$, is a spidey, i.e., is either P_2 or P_4. Since T is not a spidey itself (hence cannot be of radius 1), this finally implies that at least one T_v is a path P_4, where $v \neq r$. Fix any such T_v and denote the father of v by u, and the sons of v by v_1, v_2, where v_2 is a leaf and v_1' denotes the leaf being the only son of v_1. Then by the induction hypothesis we may find a locally irregular 3-edge colouring of a subtree T'' induced in T by all vertices except for v_1 and v_1' unless T'' is an odd length path, or a locally irregular 2-edge colouring of $T'' - v_2$ otherwise. In the first of these cases we complete the locally irregular 3-edge colouring of T by painting the edges v_1v_1' and vv_1 with a colour which does not appear on any of the remaining two edges incident with v. In the second one, it is sufficient to paint all the remaining edges, v_1v_1', vv_1 and vv_2, with the same colour as uv. Since then $d(v) = 3$ and $d(u) = 2$, the obtained 3-edge colouring shall also be locally irregular. ∎

It is worth mentioning that the result for trees above is sharp, i.e., there are infinitely many trees which require 3 colours. As an example, let us consider the following class of graphs. First, join two vertices u and v by an edge. Then choose any four paths of even lengths (≥ 2), say P_1, P_2, P_3, P_4, and identify one of the ends for each of P_1, P_2 with u, and finally identify one of the ends for each of P_3, P_4 with v. It is easy to verify that for neither of such trees there exists a locally irregular 2-edge colouring. Numerous more complex examples might also be constructed. Also many examples of graphs supporting Conjecture 3.4 might be derived from the result on products of graphs below. Let us recall that given two graphs, G, H, their *cartesian product* $G \Box H$ is defined as the graph with vertex set $V(G) \times V(H)$, where two vertices $(u_1, v_1), (u_2, v_2) \in V(G) \times V(H)$ are joined by an edge in $G \Box H$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$.

Theorem 3.10 *Suppose there exist a locally irregular k-edge colouring of a graph G and a locally irregular l-edge colouring of a graph H. Then there exists a locally irregular edge colouring of $G \Box H$ with at most* $\max\{k, l\}$ *colours.*

Proof. Let $c_1 : E(G) \to \{1, 2, \ldots, k\}$ be a locally irregular k-edge colouring of a graph G and let $c_2 : E(H) \to \{1, 2, \ldots, l\}$ be a locally irregular l-edge colouring of a graph H. We define the edge colouring c of $G\Box H$ as follows. Given an edge e with the end-vertices $(u_1, v_1), (u_2, v_2)$, we set $c(e) = c_2(v_1v_2)$ if $u_1 = u_2$ or $c(e) = c_1(u_1u_2)$ otherwise. Surely c uses at most $\max\{k, l\}$ colours. To show that it is locally irregular, without the loss of generality, let us suppose that $c(e) = 1$ where e is an edge of $G\Box H$ with the end-vertices $(u, v_1), (u, v_2)$, hence $v_1v_2 \in E(H)$ and $c_2(v_1v_2) = 1$. The set of neighbours of the vertex (u, v_1) in $G\Box H$ such that the edges joining them with (u, v_1) are coloured with 1 consists of two disjoint subsets, namely

$$\begin{aligned} N_1^{(G)} &:= \left\{(u', v_1) : uu' \in E(G) \wedge c_1(uu') = 1\right\} \quad \text{and} \\ N_1^{(H)} &:= \left\{(u, v_1') : v_1v_1' \in E(H) \wedge c_2(v_1v_1') = 1\right\}. \end{aligned}$$

Analogously, the set of neighbours of the vertex (u, v_2) in $G\Box H$ such that the edges joining them with (u, v_2) are coloured with 1 consists of the following two disjoint subsets:

$$\begin{aligned} N_2^{(G)} &:= \left\{(u', v_2) : uu' \in E(G) \wedge c_1(uu') = 1\right\} \quad \text{and} \\ N_2^{(H)} &:= \left\{(u, v_2') : v_2v_2' \in E(H) \wedge c_2(v_2v_2') = 1\right\}. \end{aligned}$$

Note that obviously, $|N_1^{(G)}| = |N_2^{(G)}|$. On the other hand, since $v_1v_2 \in E(H)$, $c_2(v_1v_2) = 1$ and c_2 is a locally irregular edge colouring of H, then $|N_1^{(H)}| \neq |N_2^{(H)}|$. The ends of our investigated edge e are thus incident with distinct numbers of edges coloured with 1 in $G\Box H$. ■

Corollary 3.11 *There exists a locally irregular* 2*-edge colouring of every hypercube* Q_n *with* $n \geq 2$ *.*

Proof. It is sufficient to prove that the required edge colourings exist for $n = 2, 3$. These can be easily found from scratch, but their existence follows also by Proposition 3.2 and Corollary 3.8, respectively. The existence of the desired edge colourings for the remaining hypercubes might then be proven inductively by Theorem 3.10, since $Q_n = Q_{n-2}\Box Q_2$ for every $n \geq 4$. ■

Using Theorem 3.10 one may also prove that e.g. grids being the cartesian products of two even length paths can be decomposed into two locally irregular subgraphs.

To close this section we shall finally prove that every connected graph which does not belong to the family $\mathfrak{T}$ and is not an odd cycle nor an odd path can be decomposed into locally irregular subgraphs. We need to prove the following lemma beforehand.

Lemma 3.12 *Let G be a connected graph whose edge set E might be partitioned into two nonempty subsets I and O (i.e., $E = I \cup O$ and $I \cap O = \emptyset$) each of which induces a connected subgraph of G. If $|O| \geq 2$, then O contains two incident edges e_1, e_2 such that $E \setminus \{e_1, e_2\}$ induces a connected graph.*

Proof. It is sufficient to prove that we may modify (if necessary) the sets O, I so that they retain all properties assumed in the hypothesis, but O consists of *exactly* two edges. For this goal, as long as $|O| \geq 3$ we keep repeating the following procedure:
Denote by G_I, G_O the graphs induced by I, O, respectively. Since G is connected, then G_O must contain a vertex, say v, incident with some edge in I. Let $G_1, \ldots, G_t$ be the components of $G_O - v$ and denote $H_i := G_O[V(G_i) \cup \{v\}]$ for $i = 1, \ldots, t$. If $t \geq 2$ and any such H_i contains at least 2 edges, then we move all the remaining edges of O except for those in H_i to I. Otherwise, we choose any edge in O incident with v and move it to I.

Since in each step we decrease the number of edges in O, we finally end up with O consisting of two incident edges. ■

Theorem 3.13 *If G is a connected graph which does not belong to $\mathfrak{T}$ and is not an odd length path nor an odd length cycle, then it can be decomposed into locally irregular subgraphs.*

Proof. Every graph which belongs to $\mathfrak{T}$ or is an odd length path or an odd length cycle shall be called an *exception*. We shall prove the theorem by induction with respect to the number of edges of G. The thesis is obvious if G contains at most 4 edges, so suppose $G = (V, E)$ is a graph of size at least 5 which is not an exception.

Note first that we may assume that G contains no vertex of degree 3 whose neighbourhood is an independent set (in G) nor any vertex of degree at least 4. Indeed, for suppose that v is such a vertex. Then if any component of $G - v$ is of size at least two, we denote its edges by O (and set $I = E \setminus O$). By Lemma 3.12 we may remove two incident edges from this component so that the remaining edges of G still induce a connected graph, say G'. By our assumption on v, G' cannot be an exception, hence by the induction hypothesis it may be decomposed into locally irregular subgraphs. Suppose then that every component of $G - v$ consists of at most one edge. If any of these components is an edge whose both ends are adjacent with v in G, then for every such edge we remove it together with one of its incident edges (incident with v). In either case, the rest of the edges of G induce a tree (even a spidey) which is not an odd length path, and hence the remaining graph may be further decomposed into locally irregular subgraphs by Theorem 3.9. We thus in particular obtain that $\Delta(G) \leq 3$ and every vertex of degree 3 must be incident with a triangle in G.

We may additionally assume that G contains no cycles of length greater than 3. To see that, suppose that C is a cycle of size at least 4 in G. If any component of the graph $G - E(C)$ is of order at least two, then analogously as above, by Lemma 3.12 (with O consisting of the edges of this component), we may remove two incident edges outside C so that the remaining edges of G induce a connected graph, say G''. Then we may decompose G'' by the induction hypothesis, unless G'' is a cycle of odd length (larger than 3) itself. It is however an easy exercise to verify that we may decompose G into locally irregular subgraphs successfully in such a case as well (it is then sufficient to 'remove' from G either a star $K_{1,3}$ or a path P_5 with an additional hanging edge appended to its middle vertex, to be left with an even path). We thus may also assume that G has circumference 3 and every of its vertices of degree 3 is incident with *exactly* one triangle (hence G is 'very similar' to the representatives of $\mathfrak{T}$, except for the assumptions on the lengths of the paths between triangles and pendant vertices).

Now if $\delta(G) = 1$, then we choose any vertex u incident with some pendant vertex and remove two edges joining u with its neighbours of possibly smallest degrees. Otherwise, G must contain a triangle with two vertices of degree 2. Then we remove the two edges incident with either of such vertices. In both cases the remaining edges induce a connected graph which cannot be an exception (since G would also be an exception otherwise). The thesis follows then by the induction hypothesis. ■

In the following section we include a list of theorems which shall be used to prove our main result on regular graphs, see Theorem 5.1 in section 5.

4 Tools

We shall use the classical tools of the probabilistic method, the Lovász Local Lemma, see e.g. [7], and the Chernoff Bound, see e.g. [20].

Theorem 4.1 (The Local Lemma; Symmetric Case) *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most D, and that $\mathbf{Pr}(A_i) \leq p$ for all $1 \leq i \leq n$. If*

$$e \cdot p \cdot (D + 1) \leq 1, \tag{4.1}$$

then $\mathbf{Pr}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

Theorem 4.2 (Chernoff Bound) *For any $0 \leq t \leq np$:*

$$\mathbf{Pr}(|\mathrm{BIN}(n,p) - np| > t) < 2e^{-\frac{t^2}{3np}},$$

where $\mathrm{BIN}(n,p)$ is the sum of n independent variables, each equal to 1 with probability p and 0 otherwise.

The following theorem from [1] has already occurred extremely useful while investigating several related problems (see also [2, 3] for similar degree theorems and their applications).

Theorem 4.3 *Suppose that for some graph $G = (V, E)$ we have chosen, for every vertex v, two integers:*

$$a_v^- \in \left[\frac{d(v)}{3} - 1, \frac{d(v)}{2}\right], \quad a_v^+ \in \left[\frac{d(v)}{2} - 1, \frac{2d(v)}{3}\right].$$

Then there exists a spanning subgraph H of G such that for every $v \in V$:

$$d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}.$$

Corollary 4.4 *Given a graph $G = (V, E)$ of minimum degree δ, a positive integer $\lambda \leq \delta/6$, and any assignment*

$$t : V \to \{0, 1, \ldots, \lambda - 1\},$$

there exists a spanning subgraph H of G such that $d_H(v) \in [\frac{d(v)}{3}, \frac{2d(v)}{3}]$ and $d_H(v) \equiv t(v) \pmod{\lambda}$ or $d_H(v) \equiv t(v) + 1 \pmod{\lambda}$ for each $v \in V$.

Proof. Note that for every $v \in V$:

$$\left\lfloor\frac{d(v)}{2}\right\rfloor - \left\lfloor\frac{d(v)}{3}\right\rfloor + 1 \geq \frac{d(v) - 1}{2} - \frac{d(v)}{3} + 1 > \frac{d(v)}{6} \geq \lambda,$$

hence, since both sides of the inequality are integers,

$$\left\lfloor\frac{d(v)}{2}\right\rfloor - \left\lfloor\frac{d(v)}{3}\right\rfloor \geq \lambda.$$

Analogously,

$$\left\lfloor\frac{2d(v)}{3}\right\rfloor - \left\lfloor\frac{d(v)}{2}\right\rfloor + 1 \geq \frac{2d(v) - 2}{3} - \frac{d(v)}{2} + 1 > \frac{d(v)}{6} \geq \lambda,$$

hence,

$$\left\lfloor\frac{2d(v)}{3}\right\rfloor - \left\lfloor\frac{d(v)}{2}\right\rfloor \geq \lambda.$$

Therefore, the sets of integers

$$\left\{\left\lfloor\frac{d(v)}{3}\right\rfloor + 1, \ldots, \left\lfloor\frac{d(v)}{2}\right\rfloor\right\} \quad \text{and} \quad \left\{\left\lfloor\frac{d(v)}{2}\right\rfloor, \ldots, \left\lfloor\frac{2d(v)}{3}\right\rfloor - 1\right\}$$

both contain all remainders modulo λ. The thesis follows then by Theorem 4.3 (it is sufficient to choose a_v^-, a_v^+ in these sets, resp., so that $a_v^-, a_v^+ \equiv t(v) \pmod{\lambda}$). ■

5 Regular Graphs

Theorem 5.1 *Every d-regular graph G with $d \geq 10^7$ can be decomposed into three locally irregular subgraphs.*

Proof. Let $G = (V, E)$ be a regular graph of degree $d \geq 10^7$. First for every vertex v we randomly and independently choose one value in $\{0, 1, \dots, \lceil d^{0.35} \rceil - 1\}$, each with equal probability, and denote it by $c_1(v)$. Then we independently repeat our drawing, i.e., again for every $v \in V$ randomly and independently we choose one value in $\{0, 1, \dots, \lceil d^{0.35} \rceil - 1\}$, each with equal probability, and denote it by $c_2(v)$. For each $v \in V$, let us denote:

$$A(v) := \{u \in N_G(v) : c_1(u) = c_1(v)\},$$

$$B(v) := \{u \in N_G(v) : c_2(u) = c_2(v)\},$$

$$C(v) := \left\{u \in N_G(v) : c_1(u) + c_2(u) \equiv c_1(v) + c_2(v) \pmod{\lceil d^{0.35} \rceil}\right\}$$

and note that:

$$D(v) := B(v) \cap C(v) = \{u \in N_G(v) : c_1(u) = c_1(v) \wedge c_2(u) = c_2(v)\}.$$

We shall first prove the following:

Claim 5.2 *With positive probability, for every vertex $v \in V$:*

$$|A(v)|, |B(v)|, |C(v)| \leq 2d^{0.65} \quad \text{and} \tag{5.1}$$

$$|D(v)| \leq 2d^{0.3} - 1. \tag{5.2}$$

Proof. For every $v \in V$, let X_v, Y_v, Z_v, T_v be the random variables of the cardinalities of the sets $A(v), B(v), C(v), D(v)$, resp., and let A_v, B_v, C_v, D_v denote the events that $X_v > 2d^{0.65}$, $Y_v > 2d^{0.65}$, $Z_v > 2d^{0.65}$ and $T_v > 2d^{0.3} - 1$, respectively. Consider any neighbour u of a given vertex v. Obviously,

$$\mathbf{Pr}\left(u \in A(v)\right) = \frac{1}{\lceil d^{0.35} \rceil} \leq \frac{1}{d^{0.35}},$$

$$\mathbf{Pr}\left(u \in B(v)\right) = \frac{1}{\lceil d^{0.35} \rceil} \leq \frac{1}{d^{0.35}}.$$

Since for every fixed $c_1(v), c_2(v)$ (and e.g. $c_1(u)$), the probability that $c_1(u) + c_2(u) \equiv c_1(v) + c_2(v) \pmod{\lceil d^{0.35} \rceil}$ equals exactly $1/\lceil d^{0.35} \rceil$, by the total probability we also obtain:

$$\mathbf{Pr}\left(u \in C(v)\right) = \frac{1}{\lceil d^{0.35} \rceil} \leq \frac{1}{d^{0.35}}.$$

Finally, since all choices are independent,

$$\mathbf{Pr}\left(u \in D(v)\right) = \left(\frac{1}{\lceil d^{0.35} \rceil}\right)^2 \leq \frac{1}{d^{0.7}}.$$

Consequently, again basing on the fact that all choices are independent, by Chernoff Bound we obtain (to be strict, we should have first written below the conditional probability with respect to some fixed value of $c_1(v)$, but since all choices are independent and we would have obtained the same upper bound regardless of the colour $c_1(v)$, then the application of the total probability would yield what follows):

$$\begin{aligned}
\mathbf{Pr}\left(A_v\right) &= \mathbf{Pr}\left(X_v > 2d^{0.65}\right) \leq \mathbf{Pr}\left(\mathrm{BIN}\left(d, \frac{1}{d^{0.35}}\right) > 2d^{0.65}\right) \\
&\leq \mathbf{Pr}\left(\left|\mathrm{BIN}\left(d, \frac{1}{d^{0.35}}\right) - d^{0.65}\right| > d^{0.65}\right) \\
&< 2e^{-\frac{d^{0.65}}{3}} \leq 2e^{-\frac{2}{7}d^{0.3}}. \qquad (5.3)
\end{aligned}$$

Analogously,

$$\mathbf{Pr}\left(B_v\right) < 2e^{-\frac{2}{7}d^{0.3}} \quad \text{and} \quad \mathbf{Pr}\left(C_v\right) < 2e^{-\frac{2}{7}d^{0.3}}. \qquad (5.4)$$

Finally, again by Chernoff Bound:

$$\begin{aligned}
\mathbf{Pr}\left(D_v\right) &= \mathbf{Pr}\left(T_v > 2d^{0.3} - 1\right) \leq \mathbf{Pr}\left(\mathrm{BIN}\left(d, \frac{1}{d^{0.7}}\right) > 2d^{0.3} - 1\right) \\
&\leq \mathbf{Pr}\left(\left|\mathrm{BIN}\left(d, \frac{1}{d^{0.7}}\right) - d^{0.3}\right| > d^{0.3} - 1\right) \\
&< 2e^{-\frac{(d^{0.3}-1)^2}{3d^{0.3}}} \leq 2e^{-\frac{\left(\sqrt{\frac{6}{7}}d^{0.3}\right)^2}{3d^{0.3}}} = 2e^{-\frac{2}{7}d^{0.3}} \qquad (5.5)
\end{aligned}$$

for $d \geq \left(\frac{1}{1-\sqrt{\frac{6}{7}}}\right)^{\frac{10}{3}} \approx 5,831$.

Since each of the events A_v, B_v, C_v and D_v depends only on the random choices for v and its adjacent vertices, then each such event corresponding to a vertex v is mutually independent of all other events corresponding to vertices v' at distance at least three from v, hence is mutually independent of all except at most $D = 3 + 4d^2$ other events. Moreover, by (5.3), (5.4) and (5.5), the probability of each of these events equals at most $2e^{-\frac{2}{7}d^{0.3}}$. In order to apply Theorem 4.1, we thus need to prove that the following inequality holds (cf. (4.1)):

$$e2e^{-\frac{2}{7}d^{0.3}}(4 + 4d^2) \leq 1. \qquad (5.6)$$

For this purpose, we shall first show that

$$f(d) := e^{\frac{1}{7}d^{0.3}} - 5d > 0 \qquad (5.7)$$

(for $d \geq 10^7$), where f is a function of d (continuous in $\mathbb{R}_+$). Note that

$$\begin{aligned} f'(d) &= \frac{0.3}{7} d^{-0.7} e^{\frac{1}{7}d^{0.3}} - 5, \\ f''(d) &= \frac{0.09}{49} d^{-1.7} e^{\frac{1}{7}d^{0.3}} \left(d^{0.3} - \frac{49}{3} \right), \end{aligned}$$

hence for $d \geq \left(\frac{49}{3}\right)^{\frac{10}{3}} \approx 11,056$, $f''(d) \geq 0$ and thus $f'(d)$ is increasing. Since at the same time, $f'(9,425,780) \approx 22 > 0$, then $f'(d) > 0$ for $d \geq 9,425,780$. The fact that $f(9,425,780) \approx 3 > 0$ thus implies that $f(d) > 0$ (in particular) for $d \geq 10^7$, hence (5.7) holds. (In fact, $9,425,780$ is the smallest integer for which f has a positive value.)

Inequality (5.7) further implies that

$$e^{\frac{2}{7}d^{0.3}} > 25d^2 \geq 6(4 + 4d^2) \geq e2(4 + 4d^2)$$

(for $d^2 \geq 24$), hence (5.6) follows. By the Local Lemma we thus obtain that

$$\mathbf{Pr} \left(\bigcap_{v \in V} \overline{A_v} \cap \overline{B_v} \cap \overline{C_v} \cap \overline{D_v} \right) > 0.$$

■

Suppose then that we have chosen the assignments c_1 and c_2 so that (5.1) and (5.2) hold for every $v \in V$. Note that since $|D(v)|$ is an integer, by (5.2) we in fact have that $|D(v)| \leq \lfloor 2d^{0.3} \rfloor - 1$. Let us temporarily remove from G all edges $uv \in E$ such that $c_1(u) = c_1(v)$ and denote the graph obtained by G'. By (5.1), we thus have:

$$\delta(G') \geq d - 2d^{0.65} = d^{0.3} \left(d^{0.7} - 2d^{0.35} \right) \geq d^{0.3} \left(36d^{0.35} + 36 \right), \tag{5.8}$$

where the last inequality, equivalent to $d^{0.7} - 38d^{0.35} - 36 \geq 0$, holds for $d^{0.35} \geq \frac{38 + \sqrt{38^2 + 4 \cdot 36}}{2}$, hence for $d \geq 34,955$. By (5.8) we thus obtain:

$$\frac{\delta(G')}{6} \geq 6d^{0.3}(d^{0.35} + 1) \geq 3\lfloor 2d^{0.3} \rfloor \lceil d^{0.35} \rceil. \tag{5.9}$$

By Corollary 4.4, we may thus find a subgraph H_1 of G' such that $d_{H_1}(v)$ has one of the two remainders modulo $\lambda = 3\lfloor 2d^{0.3} \rfloor \lceil d^{0.35} \rceil$, namely

$$d_{H_1}(v) \equiv 3\lfloor 2d^{0.3} \rfloor c_1(v), 3\lfloor 2d^{0.3} \rfloor c_1(v) + 1 \pmod{3\lfloor 2d^{0.3} \rfloor \lceil d^{0.35} \rceil} \tag{5.10}$$

for every $v \in V$, and

$$\Delta(H_1) \leq \frac{2\Delta(G')}{3} \leq \frac{2d}{3}. \tag{5.11}$$

We paint the edges of H_1 with colour 1. By (5.10), $d_{H_1}(u) \neq d_{H_1}(v)$ if $c_1(u) \neq c_1(v)$, what is fulfilled for every edge $uv \in E(G')$, hence also for every $uv \in E(H_1)$, since $H_1 \subset G'$. The graph H_1 is thus locally irregular.

Let G_1 be the graph obtained from G by removing all (already painted) edges of H_1, i.e., $G_1 = G - E(H_1)$. By (5.11),

$$\delta(G_1) \geq \frac{d}{3}. \tag{5.12}$$

Let us (again temporarily) remove from G_1 all edges uv such that $c_2(u) = c_2(v)$ or $c_1(u) + c_2(u) \equiv c_1(v) + c_2(v) \pmod{\lceil d^{0.35} \rceil}$, and denote the graph obtained by G''. By (5.1) and (5.12),

$$\delta(G'') \geq \frac{d}{3} - 4d^{0.65} = \frac{d^{0.3}}{3}\left(d^{0.7} - 12d^{0.35}\right) \geq \frac{d^{0.3}}{3}\left(108d^{0.35} + 108\right), \tag{5.13}$$

where the last inequality, equivalent to $d^{0.7} - 120d^{0.35} - 108 \geq 0$, holds for $d^{0.35} \geq \frac{120+\sqrt{120^2+4\cdot 108}}{2}$, hence for $d \geq 890,679$. By (5.13) we thus obtain:

$$\frac{\delta(G'')}{6} \geq 6d^{0.3}(d^{0.35}+1) \geq 3\lfloor 2d^{0.3} \rfloor \lceil d^{0.35} \rceil. \tag{5.14}$$

Let C be the subgraph induced by these edges uv of G_1 for which $c_1(u) + c_2(u) \equiv c_1(v) + c_2(v) \pmod{\lceil d^{0.35} \rceil}$. Note that C and G'' are edge-disjoint. For every $v \in V$, denote by

$$c_v := d_C(v) = |C(v) \cap N_{G_1}(v)| \tag{5.15}$$

the number of edges uv incident with v in G_1 such that $c_1(u) + c_2(u) \equiv c_1(v) + c_2(v) \pmod{\lceil d^{0.35} \rceil}$. Consider the subgraph D induced by these edges uv of G_1 for which $c_1(u) = c_1(v)$ and $c_2(u) = c_2(v)$. Note that $D \subset C$. By (5.2),

$$\Delta(D) \leq \lfloor 2d^{0.3} \rfloor - 1,$$

and hence, we may (greedily) find a *proper* vertex colouring

$$h : V \to \{0, 1, \ldots, \lfloor 2d^{0.3} \rfloor - 1\}$$

of D, where we take e.g. $h(v) = 0$ if v is not an end of an edge of D. By Corollary 4.4 and (5.14), we may find a subgraph H_2 of G'' such that

$$\begin{aligned} d_{H_2}(v) \;\; \equiv \;\; & 3\lfloor 2d^{0.3} \rfloor c_2(v) + 3h(v) - c_v, \\ & 3\lfloor 2d^{0.3} \rfloor c_2(v) + 3h(v) - c_v + 1 \pmod{3\lfloor 2d^{0.3} \rfloor \lceil d^{0.35} \rceil} \end{aligned} \tag{5.16}$$

for every $v \in V$. Then we colour the edges of H_2 and C with colour 2, while the remaining edges of G_1 with colour 3. Denote the graphs induced by the edges coloured with 2 and 3 by H_2' and H_3', resp., i.e., $H_2' = H_2 \cup C$ and $H_3' = G - (E(H_1) \cup E(H_2'))$. Then, since H_2 and C are edge-disjoint, by (5.15) and (5.16),

$$\begin{aligned} d_{H_2'}(v) \;\; \equiv \;\; & 3\lfloor 2d^{0.3} \rfloor c_2(v) + 3h(v), \\ & 3\lfloor 2d^{0.3} \rfloor c_2(v) + 3h(v) + 1 \pmod{3\lfloor 2d^{0.3} \rfloor \lceil d^{0.35} \rceil} \end{aligned} \tag{5.17}$$

for every $v \in V$. Therefore, $d_{H_2'}(u) \neq d_{H_2'}(v)$ if $c_2(u) \neq c_2(v)$ or $h(u) \neq h(v)$. The latter of these two conditions is obviously fulfilled for every edge $uv \in E(D)$ by the definition of h. On the other hand, $c_2(u) \neq c_2(v)$ for the remaining edges of H_2', by the definitions of C and G'' (where $H_2 \subset G''$). The subgraph of G coloured with 2 is thus locally irregular. By (5.10) and (5.17),

$$\begin{aligned} d_{H_1}(v) + d_{H_2'}(v) \equiv\ & 3\lfloor 2d^{0.3}\rfloor(c_1(v)+c_2(v)) + 3h(v), \\ & 3\lfloor 2d^{0.3}\rfloor(c_1(v)+c_2(v)) + 3h(v) + 1, \\ & 3\lfloor 2d^{0.3}\rfloor(c_1(v)+c_2(v)) + 3h(v) + 2 \pmod{3\lfloor 2d^{0.3}\rfloor\lceil d^{0.35}\rceil} \end{aligned} \tag{5.18}$$

for every $v \in V$. Since G is d-regular, and hence $d_{H_3'}(v) = d - d_{H_1}(v) + d_{H_2'}(v)$ for every $v \in V$, then by (5.18), $d_{H_3'}(u) \neq d_{H_3'}(v)$ if $c_1(u)+c_2(u) \not\equiv c_1(v) + c_2(v) \pmod{\lceil d^{0.35}\rceil}$ or $h(u) \neq h(v)$. However, by our construction all edges $uv \in E \smallsetminus E(H_1)$ with $c_1(u)+c_2(u) \equiv c_1(v)+c_2(v) \pmod{\lceil d^{0.35}\rceil}$ (i.e., the edges of C) were painted with colour 2. The graph H_3' is thus locally irregular too. ∎

6 Concluding Remarks

Note that by Theorem 5.1 and Observation 2.7 we immediately obtain the following corollary.

Corollary 6.1 *There exists a locally irregular 3-total colouring of every d-regular graph G with $d \geq 10^7$.*

By Corollary 4.4 we also directly obtain the following observation.

Corollary 6.2 *If $G = (V, E)$ is a d-regular graph with $12\chi \leq d$, where χ is the chromatic number of G, then there exists its locally irregular 2-edge colouring.*

Proof. Let $t : V \to \{0, 2, 4, \ldots, 2\chi - 2\}$ be a proper vertex colouring of G. Denote $\lambda := 2\chi$. Since then $\lambda \leq d/6$, by Corollary 4.4 there exists a spanning subgraph H of G such that $d_H(v) \equiv t(v) \pmod{\lambda}$ or $d_H(v) \equiv t(v) + 1 \pmod{\lambda}$ for each $v \in V$. Then for every edge $uv \in E$, $d_H(u) \neq d_H(v)$, hence also $d - d_H(u) \neq d - d_H(v)$. The graphs H and $G - E(H)$ thus make up a decomposition of G into two locally irregular subgraphs. ∎

Analogously as the authors of [3] we thus may derive from Corollary 6.2 another conclusion supporting Conjectures 2.3 and 3.4.

Corollary 6.3 *There exists a constant d_0 such that if G_d is a random d-regular graph (sampled uniformly from the family of all d-regular graphs of order n) for some constant $d > d_0$, then asymptotically almost surely it can be decomposed into 2 locally irregular subgraphs.*

Proof. By the result of Frieze and Łuczak [12], there exists d'_0 such that if $d > d'_0$ is a constant (and $d = o(n^\theta)$ for some constant $\theta < 1/3$), then

$$\chi(G_d) \leq \frac{d}{2\ln d}\left(1 + \frac{32\ln\ln d}{\ln d}\right) \tag{6.1}$$

with probability going to 1 as $n \to \infty$. This means that if $d > d'_0$, then asymptotically almost surely inequality (6.1) holds. Since for d sufficiently large, i.e., for $d > d''_0$, where d''_0 is some (other) constant, inequality (6.1) implies that $\chi(G_d) \leq \frac{d}{12}$, the thesis follows by Corollary 6.2 with $d_0 := \max\{d'_0, d''_0\}$. ■

Obviously we are not (yet) able to prove Conjectures 2.3 and 3.4. Using a generalization of our approach applied in the proof of Theorem 5.1 we however believe to be able to answer affirmatively to the following question, which is a weaker version of Conjecture 3.4. If correct, the proof of this fact shall be significantly more complex than the one of Theorem 5.1, and the threshold for δ much larger than 10^7.

Conjecture 6.4 *There is a constant D_0 such that if G is any graph of minimum degree $\delta \geq D_0$, then it can be decomposed into 3 locally irregular subgraphs.*

Note that Conjecture 3.4, if proven, implies the one above with $D_0 = 3$.

References

[1] L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, A. Thomason, Vertex-Colouring Edge-Weightings, Combinatorica 27(1) (2007) 1–12.

[2] L. Addario-Berry, R.E.L. Aldred, K. Dalal, B.A. Reed, Vertex colouring edge partitions, J. Combin. Theory Ser. B 94(2) (2005) 237–244.

[3] L. Addario-Berry, K. Dalal, B.A. Reed, Degree Constrained Subgraphs, Discrete Appl. Math. 156(7) (2008) 1168–1174.

[4] M. Aigner, E Triesch, Irregular assignments of trees and forests., SIAM J. Discrete Math. 3 (1990) 439–449.

[5] Y. Alavi, A.J. Boals, G. Chartrand, P. Erdős, O.R. Oellermann, K-path irregular graphs, Congr. Numer. 65 (1988) 201–210.

[6] Y. Alavi, G. Chartrand, F.R.K. Chung, P. Erdős, R.L. Graham, O.R. Oellermann, Highly Irregular Graphs, J. Graph Theory 11(2) (1987) 235–249.

[7] N. Alon, J.H. Spencer, The Probabilistic Method, 2nd edition, Wiley, New York, 2000.

[8] G. Chartrand, P. Erdős, O.R. Oellermann, How to Define an Irregular Graph, College Math. J. 19(1) (1988) 36–42.

[9] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Congr. Numer. 64 (1988) 197–210.

[10] R. Diestel, Graph Theory, Electronic Edition 2005, Springer, New York, 2005.

[11] A. Frieze, R.J. Gould, M. Karoński, F. Pfender, On Graph Irregularity Strength, J. Graph Theory 41(2) (2002) 120–137.

[12] A.M. Frieze, T. Łuczak, On the independance and chromatic numbers of random regular graphs, J. Combin. Theory Ser. B 54 (1992) 123–132.

[13] F. Havet, N. Paramaguru, R. Sampathkumar, Detection number of bipartite graphs and cubic graphs, Rapport de recherche RR- 8115, INRIA, October 2012.

[14] M. Kalkowski, A note on 1,2-Conjecture, to appear in Elect. Journ. of Comb.

[15] M. Kalkowski, M. Karoński, F. Pfender, A new upper bound for the irregularity strength of graphs, SIAM J. Discrete Math. 25(3) (2011) 1319–1321.

[16] M. Kalkowski, M. Karoński, F. Pfender, Vertex-coloring edge-weightings: Towards the 1-2-3 conjecture, J. Combin. Theory Ser. B 100 (2010) 347–349.

[17] M. Karoński, T. Łuczak, A. Thomason, Edge weights and vertex colours, J. Combin. Theory Ser. B 91 (2004) 151–157.

[18] J. Lehel, Facts and quests on degree irregular assignments, Graph Theory, Combinatorics and Applications, Willey, New York, 1991, 765–782.

[19] P. Majerski, J. Przybyło, On the irregularity strength of dense graphs, submitted.

[20] M. Molloy, B. Reed, Graph Colouring and the Probabilistic Method, Springer, Berlin, 2002.

[21] T. Nierhoff, A tight bound on the irregularity strength of graphs, SIAM J. Discrete Math. 13 (2000) 313–323.

[22] J. Przybyło, A note on neighbour-distinguishing regular graphs total-weighting, Electron. J. Combin. 15 (2008) ♯N35.

[23] J. Przybyło, M. Woźniak, On a 1,2 Conjecture, Discrete Math. Theor. Comput. Sci. 12:1 (2010) 101–108.

[24] T. Wang, Q. Yu, On vertex-coloring 13-edge-weighting, Front. Math. China 3 (2008), 1–7.