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Jakub PRZYBYŁO

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Abstract. Let $G = (V, E)$ be a graph of order n, and let $c : V \rightarrow \{1, 2, \dots, k\}$ be a not necessarily proper edge colouring. The *weight*, or the *weighted degree*, of $v \in V$ is then defined as $w(v) = \sum_{u \in N(v)} c(vu)$. The colouring c is said to be irregular if $w(u) \neq w(v)$ for every two distinct vertices $u, v \in V$. The smallest k for which such a colouring exists is called the *irregularity strength* of a graph, denoted by $s(G)$. It has been proven that $s(G) = O(\frac{n}{\delta})$. A very interesting modification, known also as the $1, 2, 3$-Conjecture, of this well studied graph invariant asserts that we wish only the *neighbours* in G to have distinct weighted degrees. Karoński, Łuczak and Thomason asked if the set of three colours $\{1, 2, 3\}$ is this time sufficient to construct a colouring c consistent with these modified requirements for each connected graph with $n \geq 3$. So far it is known that the set $\{1, 2, 3, 4, 5\}$ suffices to achieve this goal.

In this paper we further develop the study of irregular colourings, and require so that the colouring c provides distinct weights for all vertices at distance at most r. The corresponding parameter is called the r-distant irregularity strength, and denoted by $s_r(G)$. Note that s_1 coincides then with the graph invariant studied by Karoński et al., while it is also justified to write $s(G) = s_\infty(G)$. We prove that for each positive integer r, $s_r(G) \leq 6\Delta^{r-1}$, and discuss that this bound is of the right magnitude. We also investigate a total version of the problem, where for the corresponding parameter we prove $\mathrm{ts}_r(G) \leq 3\Delta^{r-1}$.

This direction of research is inspired by the concept of distant chromatic numbers. As appeared, the results obtained are also strongly related with the study on the Moore bound.

1. Introduction

Let $G = (V, E)$ be a simple, undirected and finite graph. As usual, we denote the neighbourhood of a given vertex $v \in V$ by $N(v)$, and its degree by $d(v)$. The minimum and the maximum degree of a graph will be denoted, resp., by δ and Δ. For a given $k \in \mathbb{N}$, let $[k] = \{1, 2, \dots, k\}$. Consider the following well known concept of an edge colouring inducing a colouring of the vertices introduced by Chartrand et al. [**9**]. Let $c : E \rightarrow [k]$ be a $[k]$-edge

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colouring, i.e., a *not necessarily proper* edge colouring with integers from $[k]$. For a vertex $v \in V$, denote by

$$w(v) := \sum_{u \in N(v)} c(vu)$$

the *weight* (or the *weighted degree*) of this vertex. We say that the colouring c is *irregular* if $w(u) \neq w(v)$ for every two distinct vertices $u, v \in V$. The smallest k for which we can find such a colouring for G is called its *irregularity strength*, and is denoted by $s(G)$. It exists iff G does not contain a component which is an isolated edge, and has at most one isolated vertex. We set $s(G) = \infty$ for the remaining graphs.

The exact value of irregularity strength is known for several families of graphs, including, e.g., the complete graphs, for which we have $s(K_n) = 3$, $n \geq 3$, [**9**], and the complete bipartite graphs, see [**9, 11, 14**]. In particular, $s(K_{r,r}) = 3$ for even r, and $s(K_{r,r}) = 4$ for odd r, $r \geq 4$. Let $n = |G|$ be the order of a graph G. In general it is known that $s(G) \leq n - 1$ for all graphs with finite irregularity strength except for K_3. It was first shown for connected graphs by Aigner and Triesch [**3**], and then in the remaining cases by Nierhoff [**21**]. This bound is tight, e.g., for stars, but can be significantly reduced in the case of graphs with "high" minimum degree. The influence of δ was investigated in several papers including, e.g., [**10, 12, 13, 16, 22, 23**]. In particular in [**16**] the following best general upper bound was proven:

$$s(G) \leq 6 \left\lceil \frac{n}{\delta} \right\rceil . \tag{1.1}$$

This direction of research was first initiated by Faudree and Lehel [**12**], who conjectured that there exists an absolute constant c such that $s(G) \leq \frac{n}{d} + c$ for all d-regular graphs with any $d \geqslant 2$. By counting the possible weights of the vertices, one can easily check that on the other hand we must have that $s(G) \geq \left\lceil \frac{n+d-1}{d} \right\rceil$ if G is d-regular.

A fascinating modification of the irregular colouring concept was introduced in [**18**] by Karoński, Łuczak and Thomason. Instead of requiring so that the weights of all the vertices are different, they asked for a $[k]$-edge colouring inducing a proper vertex colouring, i.e., such that the weights of each two *neighbours* (vertices joined by an edge) are distinct in a graph. Though initially they even were not able to prove that there were any absolute constant K so that a desired $[K]$-edge colouring existed for each graph G (without isolated edges), they conjectured that indeed, even $K = 3$ would suffice for any graph. The existence of such constant K was first proven by Addario-Berry et al. [**1**] by establishing that the smallest such K is at most 30. It was improved to $K \leq 16$ by Addario-Berry, Dalal and Reed [**2**], and then by Kalkowski, Karoński and Pfender [**17**], who showed recently that it is enough to use colours from the set $\{1, 2, 3, 4, 5\}$.

A completely different relaxation in the irregular colouring problem was introduced later by Bača et al. [**5**]. Let $c : E \cup V \to [k]$ be a $[k]$-*total* *colouring*, and let for a given vertex $v \in V$,

$$t(v) := c(v) + w(v)$$

denote its *total weight*. The total colouring c is said to be irregular if $t(u) \neq t(v)$ for every two distinct vertices $u, v \in V$. The smallest k for which there exists such a total colouring for a graph G is called its *total (vertex) irregularity strength*, and is denoted by $\text{tvs}(G)$. One can easily see that $\text{tvs}(G) \leq s(G)$, hence this parameter is bounded from above by $n-1$. A few improvements of this bound are included in [**5**]. On the other hand, for d-regular graphs we must have $\text{tvs}(G) \geq \left\lceil \frac{n+d}{d+1} \right\rceil$. The best (so far) upper bound dependent on the minimum degree of a graph is due to Anholcer, Kalkowski and Przybyło, who proved in [**4**] that

$$\text{tvs}(G) \leq 3 \left\lceil \frac{n}{\delta} \right\rceil + 1. \tag{1.2}$$

The following hybrid of the two concepts above was introduced in [**24**] by Przybyło and Woźniak. For a given graph G, find the smallest k so that there exists its $[k]$-total colouring inducing a proper colouring of the vertices, i.e., such that $t(u) \neq t(v)$ for every edge uv of G. The authors also posed a conjecture that the minimum absolute constant K for which there exists such a colouring for each graph G is at most 2, and proved that $K \leq 11$. This was then strongly improved by Kalkowski [**15**], who showed that for any graph G it is enough to use colours from the set $\{1, 2, 3\}$ on the edges, and 1 or 2 for the vertices. In fact a very clever and simple algorithm he invented was the cornerstone which then led to establishing best upper bounds (including inequalities (1.1) and (1.2)) in all four problems discussed above.

In this paper we further develop the study of irregular colourings. This time we ask so that the induced colouring of the vertices is not only proper, but also distinguishes the vertices at distance at most r, where $r \geq 1$ is some fixed integer, see the next section. In section 3 we discuss some examples and a relation between our research and the study on the Moore bound. Then we establish two upper bounds for both, the total (section 4) and the edge version of the problem (section 5). The proofs of these are based on the algorithmic approach inspired by the one invented by Kalkowski [**15**], and developed in [**16**] by Kalkowski, Karoński and Pfender. It is worth noting that the parameters defined in our paper unify in some sense all graph invariants discussed above (see section 2). Also a motivation for introducing this new concept are the studies on distant chromatic numbers (or the chromatic number of powers of graphs), see, e.g., a survey [**19**].

2. Distant irregularity strength

Let u, v be vertices of a graph G, and denote by $d(u, v)$ the *distance* of u and v in G, i.e., the length of the shortest path between u and v (hence

$d(v,v)=0$). If there is no such path, set $d(u,v)=\infty$. The greatest distance between any two vertices in G is the *diameter* of G, denoted by $\mathrm{diam}(G)$.

Let r be an integer, $r \geq 1$. The smallest k for which there exists a $[k]$-edge colouring of G such that:

$$w(u) \neq w(v) \text{ for every pair of vertices } u,v \text{ with } 1 \leq d(u,v) \leq r$$

will be called the *r-distant irregularity strength* of G, and denoted by $s_r(G)$. In other words, we require so that the weight of each vertex is distinguished from the weights of all other vertices at radius r (distance at most r from it). This parameter is well defined for all graphs with no independent edges. Set $s_r(G)=\infty$ for the remaining ones. Analogously, the smallest k for which there exists a $[k]$-total colouring of G such that:

$$t(u) \neq t(v) \text{ for every pair of vertices } u,v \text{ with } 1 \leq d(u,v) \leq r$$

will be called the *r-distant total irregularity strength* of G, and denoted by $\mathrm{ts}_r(G)$.

Note that s_1 and ts_1 simply denotes, resp., the second and the fourth graph invariant discussed in the introduction, hence for each graph G, $\mathrm{ts}_1(G) \leq 3$, see [**4**], and $s_1(G) \leq 5$ (if G has no K_2-component), see [**17**]. On the other hand, since the distance between any two distinct vertices in a graph is either a positive integer or is equal to ∞, it is also justified to set $s_\infty(G) := s(G)$ and $\mathrm{ts}_\infty(G) := \mathrm{tvs}(G)$. Then for each graph G:

$$s_1(G) \leq s_2(G) \leq \ldots \leq s_\infty(G).$$

3. Lower bounds and the Moore bound

Let r be a positive integer. Since for $r=1$ these new parameters are well studied, and bounded by a constant, suppose that $r \geq 2$. Then the situation changes already for $r=2$, since, e.g., $s_2(K_{1,n-1}) = s_\infty(K_{1,n-1}) = n-1$ and $\mathrm{ts}_2(K_{1,n-1}) = \mathrm{ts}_\infty(K_{1,n-1}) = \lceil \frac{n}{2} \rceil$ (see [**5**]). In general, if $\mathrm{diam}(G)=r$, then $s_r(G)=s_\infty(G)$ and $\mathrm{ts}_r(G)=\mathrm{ts}_\infty(G)$. In fact examples of graphs with relatively high distant irregularity strength are provided by constructions of "large" graphs with bounded diameter (and maximum degree).

Let $n_{\Delta,D}$ denote the largest possible number of vertices of a graph with maximum degree Δ and diameter D. It is known that this number is bounded from above by the following one:

$$M_{\Delta,D} := 1 + \Delta + \Delta(\Delta-1) + \ldots + \Delta(\Delta-1)^{D-1},$$

which is called the *Moore bound*. Note that in all cases, $M_{\Delta,D} \leq 1 + \Delta^D$. There are very few examples (or families of examples) of graphs for which $n_{\Delta,D} = M_{\Delta,D}$, see e.g. a survey by Miller and Širáň [**20**]. A lower bound for $n_{\Delta,D}$ might be obtained by analyzing the so called *undirected de Bruijn graph* of type (t,k), whose vertex set is formed by all sequences of length k, the entries of which are taken from a fixed alphabet consisting of t distinct letters, and two vertices $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ are joined by an edge if either $a_i = b_{i+1}$ for $1 \leq i \leq k-1$, or if $a_{i+1} = b_i$, for $1 \leq i \leq k-1$. Such a graph has order t^k, (maximum) degree $\Delta = 2t$ and diameter $D = k$, hence we obtain

$$n_{\Delta,D} \geq \left(\frac{\Delta}{2}\right)^D. \tag{3.1}$$

At the same time if we try to calculate the D-distant irregularity strength of this graph, say H, then we must have at least as many sums available, as there are vertices in H, hence

$$s_D(H) \geq \frac{n}{\Delta} \geq \frac{1}{2^D}\Delta^{D-1}. \tag{3.2}$$

Analogously, for the distant total irregularity strength of graphs we obtain a somewhat smaller bound, $\mathrm{ts}_D(H) \geq \frac{1}{2^D}\frac{\Delta^D}{\Delta+1}$. These can be slightly improved, since H is a regular graph. A more significant improvement might be obtained in some cases by the result of Canale and Gómez [**8**], who exhibited for an infinite set of values of Δ, families of graphs showing that $n_{\Delta,D} \geq \left(\frac{\Delta}{1.57}\right)^D$ for D congruent with -1, 0, or 1 (mod 6). In fact Bollobás conjectured [**6**] that for each $\varepsilon > 0$, it should be the case that

$$n_{\Delta,D} \geq (1-\varepsilon)\Delta^D. \tag{3.3}$$

if Δ and D are sufficiently large. He also proved [**7**] that for D going to infinity and fixed Δ there exist graphs with orders asymptotically equivalent to $M_{\Delta,D}$.

The bound (3.3), if true, shows that the general upper bound for the distant irregularity strength of a graph should essentially equal at least Δ^{r-1}. The reasoning here is the same as in the case of inequality (3.2). In the following two sections we provide the upper bounds of the stated magnitude, i.e, we prove that $\mathrm{ts}_r(G) \leq 3\Delta^{r-1}$ and $s_r(G) \leq 6\Delta^{r-1}$ for each graph G (without K_2-components). Note that this exemplifies that "large" δ does not (significantly) help in decreasing the upper bound for the parameters in question, as it was in the case of, e.g., irregularity strength, see inequality (1.1).

4. Upper bound for the r-distant total irregularity strength

For a given graph $G = (V, E)$ and $v \in V$, denote by $N^r(v)$ the set of distinct from v vertices which are at distance at most r from v in G. Note that

$$|N^r(v)| \leq d(v) + d(v)(\Delta-1) + \ldots + d(v)(\Delta-1)^{r-1} \leq d(v)\frac{M_{\Delta,r}-1}{\Delta}. \tag{4.1}$$

Let $v_1, v_2, \ldots, v_n$ be any fixed ordering of the vertices of G. For a given vertex v_i of this sequence, an edge v_iv_j will be called a *backward edge* of v_i, if $j < i$, or a *forward edge* of v_i, if $j > i$. We also denote by $d_+(v_i)$ the number of the *forward neighbours* of v_i, i.e., the vertices $v_j \in N(v_i)$ with $j > i$.

THEOREM 1. *Let G be a graph with maximum degree $\Delta \geq 1$, and let r be an integer, $r \geq 1$. Then*

$$\mathrm{ts}_r(G) \leq 3\frac{M_{\Delta,r}-1}{\Delta} \leq 3\Delta^{r-1}.$$

PROOF. Let $G = (V, E)$, and denote $M = \frac{M_{\Delta,r}-1}{\Delta}$, hence by (4.1), for each vertex $v \in V$,

$$|N^r(v)| \leq d(v)M. \tag{4.2}$$

Let us order the vertices of the graph G into any sequence $v_1, v_2, \ldots, v_n$. Initially we assign colours $M+1$ to all the edges and colour 1 to all the vertices of G. These will be modified, and a desired colouring is going to be constructed in n steps, each corresponding to a consecutive vertex of the sequence. For this purpose we shall associate with each vertex v_i two quantities, $t_f(v_i)$, which will be the final total weight of a given vertex and will not change once fixed, and $t_c(v_i)$, which will stand for a contemporary total weight of v_i and may change in each step of the algorithm. Thus initially $t_c(v_i) = (M+1)d(v_i) + 1$ for each i and none of the values $t_f(v_i)$ is fixed.

During the construction we will allow at most two modifications for each edge, where their colours will always remain in the range $\{1, \ldots, 3M\}$. The colours of the vertices will be fixed at the end of the construction and will belong to the set $\{1, \ldots, M+1\}$. So that the later is possible, we require that if only $t_f(v_j)$ is fixed for some j, then at each step of the construction (and especially at the end),

$$t_f(v_j) - M \leq t_c(v_j) \leq t_f(v_j). \tag{4.3}$$

Finally, in each, say i-th, step we wish to fix the value of $t_f(v_i)$, which is distinct from the previously fixed $t_f(v_j)$ of all the vertices v_j (with $j < i$) which are at distance at most r from v_i in G. Note that there are at most

$$|N^r(v_i)| - d_+(v_i) \leq Md(v_i) - d_+(v_i) \tag{4.4}$$

such vertices in G.

Assume now that we are about to perform the i-th step and that so far all our requirements have been fulfilled. We allow ourselves to add any number from the set $\{0, \ldots, M-1\}$ to the weight of any forward edge of v_i, and for any backward edge v_jv_i of v_i, we admit adding to it any integer (from the set $\{-M, \ldots, M\}$) that will not spoil the inequality (4.3) for v_j. We have then exactly $M+1$ (counting in the one by "0") possible modifications per each backward and M modifications per each forward edge of v_i, hence in total at least $d(v_i)M - d_+(v_i) + 1$ potential total weights for v_i. By (4.4) however, we must distinguish v_i from at most $d(v_i)M - d_+(v_i)$ vertices (we do not care right now about the forward neighbours of v_i which will be dealt with later in the construction). Therefore there exists an integer t^* among these potential total weights for v_i such that $t^* \neq t_f(v_j)$ for all vertices $v_j \in N^r(v_i)$ with fixed $t_f(v_j)$ (hence with $j < i$). Then we set $t_f(v_i) = t^*$ and perform these among the allowed additions and subtractions on the edges incident with v_i after which the total weight of v_i will be equal to t^*.

It is obvious that the condition (4.3) holds true during all the algorithm. It remains to comment on our requirement that in each step every edge colour should belong to the set $\{1,\dots,3M\}$. Note then that initially any given edge e had colour $M+1$. This could be modified only twice, once for e being a forward edge, when we were possibly adding a number from $\{0,\dots,M-1\}$ to it, and second time for e being a backward edge, when it may have been changed by some integer from the set $\{-M,\dots,M\}$.

Finally, after the last n-th step, we add to the colour of every vertex v_i (each equal to 1) the integer $x = t_f(v_i) - t_c(v_i)$, which by (4.3) belongs to the set $\{0,\dots,M\}$. This way we assure that the final colour of every vertex belongs to the set $\{1,\dots,M+1\}$. Since all $t_f(v_i)$ at distance at most r are distinct, this finishes the proof. □

5. Upper bound for the r-distant irregularity strength

THEOREM 2. *Let G be a graph without isolated edges, and with maximum degree $\Delta \geq 1$, and let r be an integer, $r \geq 1$. Then*

$$s_r(G) \leq 6\frac{M_{\Delta,r}-1}{\Delta} \leq 6\Delta^{r-1}.$$

PROOF. Since isolated edges are forbidden in our graph, we in fact have $\Delta \geq 2$. Assume that $r \geq 2$. As mentioned in the introduction, the case of $r = 1$ has already been proven by Kalkowski, Karoński and Pfender [**17**]. We may also assume that $G = (V, E)$ is connected (in each component of a graph, the maximum degree is at most Δ).

Let us order the vertices of the graph G into a sequence $v_1, v_2, \dots, v_n$ such that each v_i with $i \leq n-1$ has a neighbour later in the order (in particular, $v_{n-1}v_n \in E$), and that $d(v_{n-1}), d(v_n) \geq 2$. Such order always exists, and may be constructed by means of a spanning tree rooted in a vertex v_n, if only we assume that G is not a star. In case of a star however, our theorem holds, since then $s_r(G) \leq s(G) \leq n-1 = \Delta$ ($r \geq 2$).

We shall use the same notation of forward and backward edges as in the previous section. Analogously as above, denote $M = \frac{M_{\Delta,r}-1}{\Delta}$, hence the inequality (4.4) from the proof of Theorem 1 holds. We initially assign colour $2M+1$ to all the edges of G. This time a desired colouring is going to be constructed in $n-1$ steps, each corresponding to a consecutive vertex of the sequence, except the last one, in which the weights of both v_{n-1} and v_n will have to be adjusted simultaneously. We associate with each vertex v_i a quantity $w_c(v_i)$, which will stand for a contemporary weight of v_i and may change in each step of the algorithm. Additionally, we will associate with each vertex v_i, except the last two, a set $W_f(v_i)$ of its two possible final weights, where (for each $i \leq n-2$)

$$W_f(v_i) \in \mathcal{W} := \{\{b, b+2M\} : 0 \leq (b \bmod 4M) \leq 2M-1\}.$$

(Note that the sets from $\mathcal{W}$ are pairwise disjoint.) Hence initially $w_c(v_i) = (2M+1)d(v_i)$ for each i and none of the sets $W_f(v_i)$ is fixed. The final weights of all the vertices will be equal to their contemporary counterparts obtained after the final step of the algorithm.

During the construction we will allow at most two modifications for each edge $e \neq x_{n-1}x_n$ ($x_{n-1}x_n$ will be dealt with separately). First when e is a forward edge (of some vertex), when we allow adding to its colour any integer from the set $\{0,\ldots,2M-1\}$, and second, for e being a backward edge, when we allow only two possible modifications, i.e., adding $2M$ or subtracting $2M$ from its colour (or doing nothing). Note that the colour of each such edge e will always remain in the range $\{1,\ldots,6M\}$. In each (except the last one), say i-th, $i \leq n-2$, step of our construction we wish to fix the set $W_f(v_i) \in \mathcal{W}$ disjoint with all the previously fixed $W_f(v_j)$ of all the vertices $v_j \in N^r(v_i)$ (with $j < i$), such that we can assure that $w_c(v_i) \in W_f(v_i)$ by performing the allowed adjustments on the forward and backward edges of v_i. Once $W_f(v_i)$ is fixed, $w_c(v_i)$ must remain in it till the end of the construction.

Assume now that we are about to perform the i-th step, $i \leq n-2$, and that so far all our requirements have been fulfilled. Note that the permissible adjustments on the edges incident with v_i, i.e., these after which $w_c(v_j)$ remains in $W_f(v_j)$ for each $j < i$, allow us to obtain

$$2d(v_i)M - d_+(v_i) + 1 > 2(d(v_i)M - d_+(v_i)) \tag{5.1}$$

different consecutive integers as weights for v_i. By inequality (4.4), at most $2(d(v_i)M - d_+(v_i))$ of these are already blocked by the elements from the sets $W_f(v_j)$ of the vertices $v_j \in N^r(v_i)$ with $j < i$. Thus we are left with at least one attainable weight w' for v_i, which certainly belongs to some set $W' \in \mathcal{W}$ disjoint with all $W_f(v_j)$ for $v_j \in N^r(v_i)$ and $j < i$. Then we apply the permissible adjustments on the edges incident with v_i so that $w_c(v_i) = w'$, and denote $W_f(v_i) = W'$.

In the last step we set the weights of v_{n-1} and v_n by modifying the colours of their incident edges. We admit using any colour from the set $\{1,\ldots,6M\}$ for the edge $v_{n-1}v_n$. For the remaining edges incident with v_{n-1} or v_n, which are their backward edges, we still admit adding or subtracting $2M$ so that the weight of every v_j with $j \leq n-2$ remains in $W_f(v_j)$. It is now enough to prove that the adjustments on the edges incident with v_{n-1} or v_n can be chosen so that for the obtained vertex weights we have:

- **(I):** $w_c(v_{n-1}) \neq w_c(v_n)$,
- **(II):** for each of the two vertices v_{n-1}, v_n, its weight is different from the weights $w_c(v_j)$ of all vertices at distance at most r from it with $j < n-1$, and additionally,
- **(III):** neither of the weights $w_c(v_{n-1})$, $w_c(v_n)$ belongs to any of the sets $W_f(v_j)$ for any $v_j \in N(v_{n-1}) \cup N(v_n)$ with $j \leq n-2$.

The condition **(III)** is necessary, since we are not able to control which of the two weights from the set $W_f(v_j)$ each $v_j \in N(v_{n-1}) \cup N(v_n)$ will obtain after the required adjustments. For each $k \in \{n-1, n\}$, conditions **(II)** and **(III)** might block at most

$$d(v_k)M - 1 + (d(v_{n-1}) - 1) + (d(v_n) - 1) < (d(v_k) + 2)M \tag{5.2}$$

possible weights for v_k attainable by any admitted modifications on the edges incident with v_{n-1} or v_n, denote these blocked integers by $B^{(k)}$. We may partition this set into $2M$ subsets, $B^{(k)} = B_1^{(k)} \cup B_2^{(k)} \cup \ldots \cup B_{2M}^{(k)}$, where for each $q \in \{1, \ldots, 2M\}$, $B_q^{(k)}$ consists of these weights from $B^{(k)}$ which require using an integer equivalent to q modulo $2M$ as a colour of the edge $v_{n-1}v_n$ (i.e., q, $q + 2M$ or $q + 4M$). Let $b_q^{(k)} = |B_q^{(k)}|$ and denote $p_q^{(k)} = \frac{b_q^{(k)}}{d(v_k)+2}$ for $q \in \{1, \ldots, 2M\}$, $k \in \{n-1, n\}$. Then there must exist $q_0 \in \{1, \ldots, 2M\}$ such that $p_{q_0}^{(n-1)} + p_{q_0}^{(n)} < 1$. Otherwise,

$$2M \leq \sum_{q=1}^{2M} \left(p_q^{(n-1)} + p_q^{(n)}\right) = \sum_{q=1}^{2M} p_q^{(n-1)} + \sum_{q=1}^{2M} p_q^{(n)},$$

hence at least one of the two sums, say the second one, on the right hand side of the equality above is at least M, but then

$$(d(v_n) + 2)M \leq \sum_{q=1}^{2M} p_q^{(n)}(d(v_n) + 2) = \sum_{q=1}^{2M} |B_q^{(k)}| = |B^{(k)}|,$$

thus we obtain a contradiction with inequality (5.2).

Note that since $(d(v_{n-1})+2), (d(v_n)+2) \geq 4$, the fact that $p_{q_0}^{(n-1)} + p_{q_0}^{(n)} < 1$ implies that at least one of the following four conditions must hold:

(i): $b_{q_0}^{(n-1)} = d(v_{n-1}) + 1$ and $b_{q_0}^{(n)} \leq d(v_n) - 2$, or
(ii): $b_{q_0}^{(n-1)} \leq d(v_{n-1})$ and $b_{q_0}^{(n)} \leq d(v_n) - 1$, or
(iii): $b_{q_0}^{(n-1)} \leq d(v_{n-1}) - 1$ and $b_{q_0}^{(n)} \leq d(v_n)$, or
(iv): $b_{q_0}^{(n-1)} \leq d(v_{n-1}) - 2$ and $b_{q_0}^{(n)} = d(v_n) + 1$.

We consider the first two of these four cases, the later two being symmetric to the others. We first try to settle the final weight of v_{n-1}. Note that if we wish to use a colour congruent to q_0 modulo $2M$ for the edge $v_{n-1}v_n$ (i.e., q_0, $q_0 + 2M$ or $q_0 + 4M$), then the admitted adjustments on the edges incident with v_{n-1} provide us with exactly $d(v_{n-1}) + 3$ possible weights for v_{n-1}.

Suppose that **(i)** is true. Then at least one (even two) of these possible weights for v_{n-1} is not blocked by conditions **(II)** or **(III)**, hence we modify (and finally fix) the colours of the edges incident with v_{n-1} to obtain this weight at v_{n-1}. Then by the allowed modifications on the remaining edges incident with v_n, i.e., all except $v_{n-1}v_n$, we may obtain $d(v_n)$ distinct weights for v_n, at least two of which are not blocked by conditions **(II)** or **(III)**. Among these at least one is different from $w_c(v_{n-1})$, and thus guarantees also the fulfillment of condition **(I)**.

Suppose then that **(ii)** holds. Then at least three of the possible weights of v_{n-1} attainable with a colour congruent to q_0 modulo $2M$ used on the edge $v_{n-1}v_n$ are not blocked by conditions **(II)** or **(III)**. Denote them by w_1, w_2 and w_3. For each of these, after the corresponding adjustments, we may analogously as above obtain a set of $d(v_n)$ distinct weights for v_n. Denote these sets, resp., by S_1, S_2 and S_3. If in any of theses sets there are at least two integers not blocked by **(II)** or **(III)**, then analogously as above, choosing one of these two weights for v_n guarantees the fulfillment of condition **(I)**. Otherwise however, since at most $d(v_n)-1$ elements from $S_1 \cup S_2 \cup S_3$ might be blocked by conditions **(II)** and **(III)**, and each of the sets S_1, S_2, S_3 forms an arithmetic progression of common difference $2M$, there must exist a common element of at least two of these sets, say $w \in S_1 \cap S_2$, which is not blocked by **(II)** or **(III)**. Moreover, since $w_1 \neq w_2$, we must have $w \neq w_1$ or $w \neq w_2$, hence performing the corresponding adjustments on the edges incident with v_{n-1} or v_n yields a desired colouring. □

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AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

E-mail address: przybylo@wms.mat.agh.edu.pl