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Cheapest $(K_q; k)$-stable graphs

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Abstract

A graph G is called $(H; k)$-*stable* if G contains a subgraph isomorphic to H ever after removing any k elements each of which is either a vertex or an edge of G. Given a cost α of every vertex and a cost β of every edge of G we define the total cost $c(G)$ of G to be $c(G) = \alpha|G| + \beta||G||$. By $\text{stab}_{(\alpha,\beta)}(H; k)$ we denote the minimum cost among the costs of all $(H; k)$ stable graphs. In the paper, for all $\alpha, \beta \geq 0$, we present the exact value of $\text{stab}_{(\alpha,\beta)}(K_q; k)$ for infinitely many k.

1 Introduction

By a word graph we mean a simple graph in which multiple edges (but not loops) are allowed. Given a graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. Furthermore, $|G| := |V(G)|$ is the order of G and $||G|| := |E(G)|$ is the size of G.

Let H be any graph and k a non-negative integer. A graph G is called $(H; k)$-*stable* if $G - S$ contains a subgraph isomorphic to H for every set $S \subset V \cup E$ with $|S| \leq k$. Given the cost $\alpha \geq 0$ of every vertex, and the cost $\beta \geq 0$ of every edge, the total cost $c(G)$ of G is defined by $c(G) = \alpha|G| + \beta||G||$. Then $\text{stab}_{(\alpha,\beta)}(H; k) = \min\{c(G) : G \text{ is } (H; k) \text{ stable}\}$ denotes the minimum cost among the costs of all $(H; k)$-vertex stable graphs.

Note that if $S \subset V$ and $\alpha = 0, \beta = 1$ then the above problem reduces to the problem of finding minimum $(H; k)$-vertex stable graphs, with the minimum cost (= minimum size) denoted by $\text{stab}(H; k)$. This problem has been investigated in several papers including [1, 2, 3, 6]. In particular, the following result was obtained

Theorem 1 ([6]) *If $q \geq 2$ and $k \geq (q-3)(q-2) - 1$, then*

$$\text{stab}(K_q; k) = (2q - 3)(k + 1).$$

Moreover, if G is a $(K_q; k)$-stable with $||G|| = (2q-3)(k+1)$ then G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

In this paper, for all $\alpha, \beta \geq 0$ and $q \geq 2$, we present the exact value of $\text{stab}_{(\alpha,\beta)}(K_q; k)$ for infinitely many k. However, we prove only a special case $\alpha = \beta = 1$. For the proof of the whole result, we refer the reader to the full version of this article [7].

It is worth mentioning that a 'clear' edge version (i.e. with $S \subset E$ and $\alpha = 0, \beta = 1$) of this problem has also been considered, see [4, 5].

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2 Main result

We start with the following lemma.

Lemma 2 *If G is $(H;k)$-stable with minimum cost, then*

$$|G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq k+1. \tag{1}$$

Moreover, if G is not a union of cliques then the inequality (1) is strong.

This lemma is completely analogous to Theorem 2 in [6]. For completness we repeat the proof from [7].

Proof of Lemma 2. Let σ be an ordering of the vertices of G. For $v \in V(G)$ let $\deg^-_\sigma(v)$ denote the number of neighbors of v that are on the left from v in ordering σ. Let S_σ denote the set of all vertices v with $\deg^-_\sigma(x) \leq \delta_H - 1$. Note that by removing from G all vertices from $V(G) \setminus S_\sigma$ we spoil all copies of H. Indeed, we can consecutively (from the right to the left) eliminate all vertices from S_σ because at each time the analized vertex has degree $\leq \delta_H - 1$ (and therefore is useless for H). Thus, since G is $(H;k)$-stable, $|G| - |S_\sigma| \geq k+1$ for each ordering σ.

Therefore, it suffices to find an ordering σ with $|S_\sigma| \geq \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$. We assume that $\delta_H \geq 2$, because for $\delta_H = 1$ each set S_σ is an independent set and the undermentioned facts are well known. Given a random ordering σ, the probability that a vertex v has at most i neighbours on its left side in the ordering σ is equal

$$Pr(\deg^-_\sigma(v) \leq i) = \frac{\binom{n}{d_G(v)+1}(i+1)(d_G(v))!(n-d_G(v)-1)!}{n!} = \frac{i+1}{d_G(v)+1}.$$

Thus,

$$Pr(v \in S_\sigma) = \frac{\delta_H}{d_G(v)+1}.$$

Hence,

$$E\left(|S_\sigma|\right) = \sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}.$$

Thus, there exists an ordering σ with the required number of vertices in S_σ. Furthermore, the equality in (1) may hold only if $|S_\sigma|$ is the same for every ordering σ (if there is a σ with $|S_\sigma| < \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$, then there is also a σ' with $|S_{\sigma'}| > \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ because the expectation is exactly that number). Now we will prove that if G is minimum $(H;k)$-stable, then this is possible only for the disjoint union of cliques.

Let C be any component of G and let $v \in V(C)$. Note that since G is a $(H;k)$-stable with minimum cost, every vertex (as well as every edge) of G is contained in some copy of H. Thus, the minimum degree of G is at least δ_H. Let $\delta = \delta_H$. Let v be an arbitrary vertex of G. Consider the following ordering σ of vertices of C:

$$v_1, v_2, ..., v_\delta, v_{\delta+1}, v_{\delta+2}, ..., v_{|C|},$$

where $v_{\delta+1} = v$ and $v_1, v_2, ..., v_\delta$ are any neighbours of v. Next consider an ordering σ'

$$v_{\delta+1}, v_1, v_2, ..., v_\delta, v_{\delta+2}, ..., v_{|C|}.$$

Note that since $|S_\sigma| = |S_{\sigma'}|$ and $v_{\delta+1} \in S_{\sigma'}$, $v_\delta \notin S_{\sigma'}$. Thus, $\deg^-_{\sigma'}(v_\delta) = \delta$. Analogously we obtain that $\deg^-_{\sigma''}(v_{\delta-1}) = \delta$ in an ordering $\sigma'' : v_\delta, v_{\delta+1}, v_1, v_2, ..., v_{\delta-1}, v_{\delta+2}, ..., v_{|C|}$, and so on. Therefore, vertices $v_1, v_2, ..., v_\delta, v_{\delta+1}$ induce a clique. Since v and its neighbours have been chosen arbitrarily, $\{v\} \cup N_G(v)$ induce a clique for each $v \in V(C)$. This implies that C is a clique. □

Theorem 3 *If $q \geq 2$ and $k \geq (q-1)(q-2)-1$, then*

$$\mathrm{stab}_{(1,1)}(K_q; k) = (2q-1)(k+1). \tag{2}$$

Moreover, if G is a $(K_q; k)$-stable with $c(G) = (2q-1)(k+1)$ then G is a disjoint union of cliques K_{2q-2} and K_{2q-1}.

Proof. Let G be a $(K_q; k)$-stable graph with minimum possible cost. By Lemma 2 we have that

$$|G| \geq (q-1) \sum_{v \in V(G)} \frac{1}{d_G(v)+1} + k + 1 \geq |G| \frac{q-1}{d_G+1} + k + 1, \text{ and so}$$
$$|G| \geq (k+1) \frac{d_G+1}{d_G-q+2},$$

where $d_G = \frac{2||G||}{|G|}$ is the average degree of G. Thus,

$$c(G) = |G| + ||G|| = |G| + \frac{d_G}{2}|G| \geq \left(1 + \frac{d_G}{2}\right)(k+1)\frac{d_G+1}{d_G-q+2}.$$

By examining the derivative of the function $f(x) = \left(1+\frac{x}{2}\right)(k+1)\frac{x+1}{x-q+2}$ we obtain that f is decreasing for $x \leq x_0$ and increasing for $x \geq x_0$ where $x_0 = q-2+\sqrt{(q-1)q}$. Note that

$$2q-3 < x_0 < 2q-2. \tag{3}$$

Therefore, the lower bound (2) can be achieved only if $d_G \in [2q-3, 2q-2]$. Indeed, otherwise $c(G) > (2q-1)(k+1)$, since $f(2q-3) = f(2q-2) = (k+1)(2q-1)$. Then the sum $\sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ is minimal if degrees of vertices of G differ as small as possible from d_G. Thus, we may assume that $d_G(v) \in \{2q-3, 2q-2\}$ for every $v \in V(G)$. Let m denote the number of vertices of G with degree equal to $2q-3$. Hence,

$$\sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-1}, \tag{4}$$

with equality if and only if $d_G(v) \in \{2q-3, 2q-2\}$ for every $v \in V(G)$. Therefore, by Lemma 2 we have

$$|G| - m\frac{q-1}{2q-2} - (|G|-m)\frac{q-1}{2q-1} \geq k+1, \text{ and so} \tag{5}$$
$$|G| \geq (k+1)\frac{2q-1}{q} + \frac{m}{2q},$$

and if equality holds, then G is a disjoint union of cliques.

Thus,

$$\begin{aligned} c(G) &= |G| + ||G|| = |G| + \frac{1}{2}(m(2q-3) + (|G|-m)(2q-2)) \\ &= |G|q - m/2 \geq (2q-1)(k+1) + m/2 - m/2 \qquad (6) \\ &= (2q-1)(k+1), \end{aligned}$$

by formula (5). Moreover, if equality holds, then G is the disjoint union of cliques K_{2q-2} and K_{2q-1}.

Now we will show that the equality in formula (6) is indeed attained for disjoint union of cliques K_{2q-2} and K_{2q-1}. Note that $aK_{2q-1} + bK_{2q-2}$ is $(K_q; aq + b(q-1) - 1)$-stable. Hence, for $k = aq + b(q-1) - 1$,

$$\begin{aligned} c(aK_{2q-1} + bK_{2q-2}) &= a(2q-1) + a\frac{(2q-1)(2q-2)}{2} + b(2q-2) + b\frac{(2q-2)(2q-3)}{2} \\ &= (2q-1)(aq + b(q-1)) = (2q-1)(k+1) \end{aligned}$$

as required.

On the other hand $(q-1)(q-2)-1$ is the Frobenious number for $\{q, q-1\}$, namely the largest integer that canot be presented in the form $aq + b(q-1)$. Thus, if $k \geq (q-1)(q-2) - 1$, then $G = aK_{2q-1} + bK_{2q-2}$ is $(K_q; k)$-stable with minimum cost. □

For arbitrary α, β we have

Theorem 4 ([7]) *Let $\alpha \geq 0$, $\beta > 0$ be real numbers and $k \geq 0$, $q \geq 2$ be integers. Let $r = \left\lfloor \sqrt{(q-1)(q-2+\frac{2\alpha}{\beta})} \right\rfloor - q$.*

1. *If $\alpha(q-1) > \beta(2q + qr + (r^2 + r - 2)/2)$, then*

$$\mathrm{stab}_{(\alpha,\beta)}(K_q; k) \geq (k+1)\frac{2q+r}{q+1+r}\left(\alpha + \beta\left(q + \frac{r-1}{2}\right)\right),$$

with equality if and only if $k = a(q+1+r) - 1$ for some positive integer a. Moreover, if G is $(H; k)$-stable with minimum cost then G is a disjoint union of cliques K_{2q+r}.

2. *If $\alpha(q-1) = \beta(2q + qr + (r^2 + r - 2)/2)$, then*

$$\mathrm{stab}_{(\alpha,\beta)}(K_q; k) \geq (k+1)\frac{2q+r}{q+1+r}\left(\alpha + \beta\left(q + \frac{r-1}{2}\right)\right),$$

with equality if and only if $k = a(q+1+r) + b(q+r) - 1$ for some positive integers a, b. Moreover, if G is $(H; k)$-stable with minimum cost then G is a disjoint union of cliques K_{2q+r} and K_{2q-1+r}.

3. *If $\alpha(q-1) < \beta(2q + qr + (r^2 + r - 2)/2)$, then*

$$\mathrm{stab}_{(\alpha,\beta)}(K_q; k) \geq (k+1)\frac{2q-1+r}{q+r}\left(\alpha + \beta\left(q + \frac{r-2}{2}\right)\right),$$

with equality if and only if $k = a(q+r) - 1$ for some positive integer a. Moreover, if G is $(H; k)$-stable with minimum cost then G is a disjoint union of cliques K_{2q-1+r}.

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