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Coverings of cubic graphs and 3-edge colorability *

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Abstract

Let $h\colon \tilde{G} \to G$ be a finite covering of 2-connected cubic (multi)graphs where G is 3-edge uncolorable. In this paper, we find conditions under which $\tilde{G}$ is 3-edge uncolorable. As the examples, we have constructed a regular 5-fold covering $f\colon \tilde{G} \to G$ of uncolorable cyclically 4-edge connected cubic graphs and a non-regular 5-fold covering $g\colon \tilde{H} \to H$ of uncolorable cyclically 6-edge connected cubic graphs.

In [11], Steffen introduced the resistance of a subcubic graph, a characteristic that measures how far is this graph from being 3-edge colorable. In this paper, we also study the relation between the resistance of the base cubic graph and the covering cubic graph.

Keywords: uncolorable cubic graph, covering of graphs, voltage permutation graph, resistance, nowhere 4-flow

Mathematics Subject Classification: 05C15, 05C10

1 Introduction

We shall consider only subcubic graphs i.e. graphs in which the degree of any vertex does not exceed three. Let $\chi'(G)$ denote the chromatic index of the subcubic graph G. The graph G is called colorable if $\chi'(G)$ is less than or equal to three, otherwise it is called uncolorable. An uncolorable cubic graph is called a snark if it is cyclically 4-edge connected and its girth is at least five.

There are known constructions that allow to produce new snarks starting from small cubic graphs and applying to them some operations (for example, via the dot product, the vertex and edge superpositions [7], Loupekine construction [3], gluing multipoles *etc.*)

The motivation of this paper is an attempt to understand whether the uncolorable cubic graphs (in particular, snarks) can be obtained via covering maps i.e., starting from an uncolorable cubic graph and lifting it via a covering map, and what are the conditions under which such lifting is

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successful (see Section 1 for the definition of covering of graphs). Intuitively, a covering map of graphs is a "regular" homomorphism of them, so the question seems to be natural. Covering of graphs are usually described via voltage graphs or permutation voltage graph construction [5].

The structure of this paper is the following. In Introduction, we define some notions and concepts from topological graph theory, such as coverings of graphs, voltage graph, voltage permutation graph, i.e. graphs enhanced with an additional structure which allow to describe coverings. For details see also [5].

In Section 2 we study general conditions under which, for a given covering of cubic (multi)graphs, the covering graph is to be uncolorable (Theorem 2.2). Theorem 2.2 relies basically on a standard procedure of gluing several copies of the same multipole in some consistent way (actually in a cyclic order) and allows to restate many other results on multiplying snarks in terms of topological graph theory. On the other hand, we provide a nonstandard procedure for obtaining uncolorable graphs by using 5-fold non-regular coverings of cubic graphs. Under certain conditions, this allows to produce a big class of cyclically 6-edge connected snarks.

In Section 3, we study coverings of cubic graphs G with respect to resistance $r(G)$, a parameter of uncolorable cubic graphs that measures how far is a given cubic graph from being 3-edge colorable. Another interesting measure of noncolorability of a bridgeless cubic graph G is its oddness, denoted by $\omega(G)$. This is the minimum number of odd cycles that are in G after removing in it a 1-factor. By definitions, we have obviously $r(G) \leq \omega(G)$. The parameters $r(G)$ and $\omega(G)$ were introduced and studied by E.Steffen in [11]. The main problem was to construct for each natural number n a cubic graph of minimum order such that $r(G) = n(\omega(G) = n$, respectively). In an equivalent form, the problem is to construct 2-connected cubic graphs G with the maximum ratio $\rho(G) = r(G)/|G|$ ($\mu(G) = \omega(G)/|G|$, respectively) or estimate these parameters asymptotically. In [6], J.Hägglund has improved the previous results of Steffen. The best known estimates of ratios $\rho(G) = r(G)/|G|$ and $\mu(G) = \omega(G)/|G|$ were given by R.Lukot'ka, E.Máčajová, J.Mazák, M.Škoviera in [8]. A good survey on measures of noncolorability of cubic graphs is the recent paper [2] where some improvement of the previous known results is also given.

In Section 3, we show that under certain conditions the resistance of a cubic graph increases when passing from the base graph G to the covering graph $\tilde{G}$ (Theorem 3.1). We supply our consideration with examples.

Finite coverings of cubic graphs were the powerful tool in proving the Heawood conjecture on the chromatic number of a closed surface. By using them, one can construct triangular embeddings of complete graphs K_n (in regular cases) or the complete graphs with a few edges removed into closed surfaces of corresponding genus. The combinatorial schemes of such triangulations were described by means of current and voltage graphs that are modeled over cubic graphs with the assignment in a finite group H.

Definition 1.1 *A surjective (continuous) map* $p\colon \tilde{S} \to S$ *of topological spaces* $\tilde{S}$ *and* S *is called a covering map(covering) if for each* $x \in S$ *there exists a neighbourhood* $U(x)$ *such that* $p^{-1}(U(x))$ *is decomposed into disjoint sum* $\bigsqcup_{i\in I} U_i$ *of sets* U_i *such that for each* $i \in I$*, where* I *is a countable set, the restriction* $p|_{U_i}\colon U_i \to U(x)$ *is a homeomorphism. Then* $\tilde{S}$ *is called the covering space and* S *the base space(or simply the base) of the covering* p.

Moreover, restricted to graphs, we also require that the covering $p\colon \tilde{G} \to G$ is a graph map. In the following, we also require that both the covering graph and the base graph of the covering $p\colon \tilde{G} \to G$

are finite and connected. In this case, the cardinal number $n = |p^{-1}(x)|$ does not depend on the choice of $x \in S$. If n is a finite number, then p is called the n-fold covering map.

Note that if the covering space $\tilde{S}$ in the covering $p\colon \tilde{S} \to S$ is not connected, then restricting p to each connected component S_i of $\tilde{S}$, we shall obtain the covering maps with the desired property. A covering map p is called *regular* if the deck transformation group H acts on $\tilde{S}$ transitively [5].

Definition 1.2 *Let $G = (V, E)$ be a connected graph. We can replace each edge $e \in E$ with the two arcs, e' and e'', joining the same pair of vertices, but with opposite directions. As a result, we shall obtain a directed graph G' with the set of arcs E'. Let A be a finite group and let $\alpha\colon E' \to A$ be a map which satisfies the following condition: for any $e \in E$, if $\alpha(e') = h \in A$, then $\alpha(e'') = h^{-1} \in A$. The pair (G, α) is called then a voltage graph and the mapping α a voltage assignment on G.*

Let G be a graph. By taking an orientation of edges of G we obtain an orgraph $\overrightarrow{G}$ with the same of edges (which also are called the arcs of $\overrightarrow{G}$). It is clear that the voltage assignment α on G is uniquely determined by its values on the arcs of $\overrightarrow{G}$. For this reason, when defining a voltage assignment $\alpha\colon E' \to A$, we indicate only the values of α on arcs from $\overrightarrow{E}$.

Definition 1.3 *The derived graph G^α is defined in the following way: $V(G^\alpha) = V \times A$ and $E(G^\alpha) = E \times A$. More precisely, if $e = (u, v)$ is an arc from u to v in $\overrightarrow{E}$, then the edge (e, g) of G^α joins the vertices (u, g) and $(v, g \cdot \alpha(e))$.*

The derived graphs G^α with the voltage assignment in a group H of order n, describe regular n-fold coverings of the graph G as follows.

Proposition 1.1 *[4] Every regular n-fold covering $f\colon \tilde{G} \to G$ of graphs with the finite deck transformation group H where G is connected and $|H| = n$ is realized by a voltage graph (G, α) with a voltage assignment in H.*

In general, coverings of graphs are described by the following

Proposition 1.2 *[4]Every n-fold covering $f\colon \tilde{G} \to G$ of graphs is realized by a permutation voltage graph with an assignment in the symmetric group Σ_n.*

Here the permutation voltage assignment is some generalization of the voltage assignment described before. Take an orientation of the graph G. A permutation voltage assignment in Σ_n is a function β, defined on the arcs of the orgraph $\overrightarrow{G}$, that assigns to each arc of $\overrightarrow{G}$ a permutation in Σ_n. The pair (G, β) is called a permutation voltage graph. As in the case of the voltage assignment, we assume that the function β is extended to the whole set of arcs of G', so that the following condition is satisfied: if $e \in \overrightarrow{G}$ and $\beta(e) = \omega \in \Sigma_n$, then $\beta(e^{-1}) = \omega^{-1}$. The derived graph associated with a permutation voltage graph (G, β) is denoted by G^β.

Let $c = (e_1, \dots, e_{k-1}, e_k)$ be an oriented path in the voltage permutation graph (G, β). We define the permutation $\beta(c) \in \Sigma_n$ as follows: $\beta(c) = \beta(e_1) \cdot \beta(e_2) \dots \beta(e_k)$. If c is an oriented cycle in G, the element $\beta(c)$ of the group Σ_n is defined up to conjugate.

In the following, we also consider graphs with semiedges (see also [?]). A multipole is a triple $M = (V; E; S)$ where $V = V(M)$ is the vertex set, $E = E(M)$ is the edge set and $S = S(M)$ is the set of semiedges. Each semiedge is incident to exactly one vertex v of M and is denoted by

(v) (the second end of semiedge contains no vertex of G); The last condition means that no loop cannot serve as a semiedge of M. Semiedges are usually grouped into pairwise disjoint connectors [7, 9]. A multipole with k semiedges is called k-pole. If $S(M) = \emptyset$, then M is simply a graph. We say that the graph M' is obtained from the $2k$-multipole M by identifying the pairs (v_i) and (u_i) of semiedges where $i = 1, \ldots, k$, if each such pair (v_i) and (u_i) is replaced with an edge $\{v_i, u_i\}$ in M'.

Let M be a multipole and let $[k] = \{1, 2, \ldots, k\}$ be a set of colors. Let $f\colon M \to [k]$ be a mapping that assigns to each $e \in E \cup S$ a color from $[k]$ in such a way that for every vertex v in M the ends incident with v (edges or semiedges) have pairwise distinct colors. Then f is called a k-edge coloring of M. Therefore if M is a cubic multipole that has a k-edge coloring, then $k \geq 3$. Moreover if M is a loopless cubic multipole, then there exists an m-edge coloring of M with $m \leq 4$. If there is a 3-edge coloring of M, we say that M is colorable, otherwise it is uncolorable.

Sometimes it is convenient to consider the colors $1, 2$, and 3 as nonzero elements of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ and redefine a 3-edge coloring of a graph or a multipole in terms of nowhere-zero flows. For convenience of the reader, below we provide some relevant information on this subject.

Let G be a (multi)graph, $\overrightarrow{G}$ an orientation of G and H be an abelian group. Under an H-flow on G we shall mean a nowhere-zero circulation $f\colon \overrightarrow{E} \to H$ [1]. Moreover under a k-flow on G we shall mean a nowhere-zero circulation f with values in the cyclic group $\mathbf{Z}_k$. We shall say that the (multi)graph G has a k-flow if such k-flow exists for some orgraph (oriented multigraph) $\overrightarrow{G}$ with the underlying (multi)graph G.

Nowhere-zero k-flows on a multipole are defined in the same way as for cubic (multi)graphs. The only difference is that any k-flow on an l-multipole M has nontrivial sources(sinks) just at the semiedges of M. We consider nowhere-zero flows on graphs and multipoles G with values in $\mathbf{Z}_2 \times \mathbf{Z}_2$. In this case, the orientation of edges (semiedges) of G is irrelevant. Recall that for cubic (multi)graphs and multipoles G the following conditions are equivalent [1]:

a) G has a 4-flow;

b) G is 3-edge colorable.

Note that if $M = (V, E, S)$ is a cubic multipole and $\varphi\colon E \cup S \to \mathbf{Z}_2 \times \mathbf{Z}_2$ is a (nowhere zero) 4-flow on M, then $\sum_{e \in S} \varphi(e) = 0$ [7].

A simple graph or a multigraph that does not have a 4-flow is called *4-snark*. Cyclically 4-edge connected uncolorable cubic graphs with girth at least 5 are called *snarks*.

Below we provide an example of uncolorable graph G and its 3-fold cover graph $\tilde{G}$ which is colorable.

Example 0. In Fig. 1, it is shown a 16-pole G' embedded into the rectangular R. Gluing together the pair of vertical sides and the pair of horizontal sides of R, we obtain a torus T. The corresponding six pairs of "vertical" semiedges (e_1 and e_2, a_1 and a_2, d_1 and d_2, b_1 and b_2, f_1 and f_2, c_1 and c_2) and the pairs of "horizontal" semiedges (s_1 and s_2, t_1 and t_2) in G are also identified. As a result, we shall obtain a graph G embedded in the torus T (in which each pair of corresponding semiedges of G' is replaced with a unique edge of G).

The snark G is one of the third powers of the Petersen graph P (via the dot product), so we simply write $G = P^3$ (see [10]).

Take the orientation of the six "vertical" edges of P^3 (i.e. a, b, c, d, e and f) from bottom to the top and an arbitrary orientation of the remaining edges. Cutting the graph P^3 along the six "vertical" edges, we shall obtain a 12-pole H which has a natural embedding in a cylinder.

Fix a natural number $n \geq 2$. Define the voltage assignment $\alpha\colon E(P^3) \to \mathbf{Z}_n$ as follows: $\alpha(h) = 1$

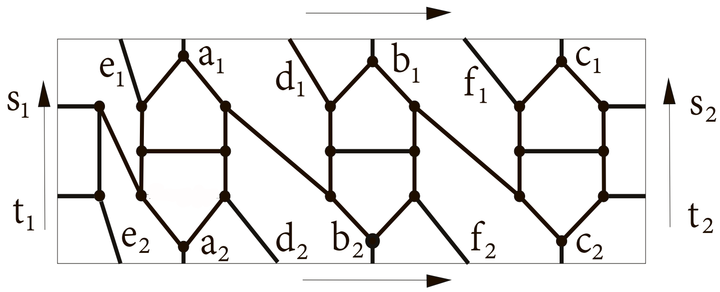

Figure 1: The 16-pole G'

if h is one of six "vertical" edges and $\alpha(h) = 0$ in the remaining cases. The voltage graph (P^3, α) defines the derived cubic graph $\widetilde{P^3}$. The corresponding n-fold covering map $p: \widetilde{P^3} \to P^3$ is cyclic. The n-fold covering of graphs can be extended to a cyclic n-fold covering $f: \tilde{T} \to T$ of tori in a natural way. For $n = 3$ the n-fold covering graph $\widetilde{P^3}$ embedded in the torus $\tilde{T}$ is pictured in Fig. 2 (here we identify also the corresponding semiedges in the pairs).

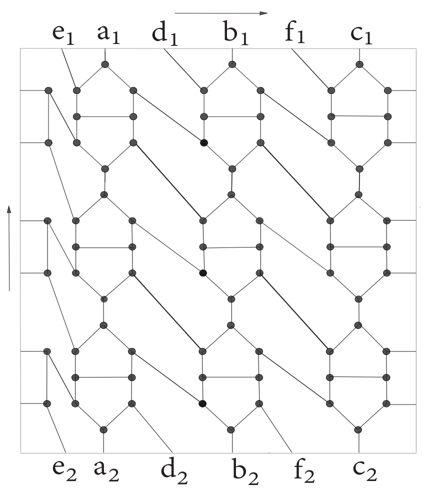

Figure 2: The 3-fold covering graph $\widetilde{P^3}$ embedded in the torus $\tilde{T}$

Note that the multipole H has a 3-edge coloring in which all six bottom semiedges receive a color x and all six top semiedges receive a color y where $x \neq y$. It follows that for any choice $n \geq 2$ the covering cubic graph $\widetilde{P^3}$ is colorable. The details of the proof are left to the reader as an easy exercise.

2 Voltage graphs and nowhere-zero flows

The following is an immediate consequence of definitions of the n-flow and a covering map.

Proposition 2.1 *Let $p{:}\,\tilde{G} \to G$ be the m-fold covering map of graphs. If G has an n-flow (where $n \geq 2$) then $\tilde{G}$ also has an n-flow.*

Proof. Let $\overrightarrow{G}$ be an orgraph with the underlying graph G and let f be an n-flow on $\overrightarrow{G}$. Note that the orientation of edges of the graph G is lifting uniquely to an orientation of edges in the covering graph $\tilde{G}$. Let $\tilde{G}'$ denote the resulting orgraph with the underlying graph $\tilde{G}$. We define the function $\overline{f}$ on $E(\tilde{G}')$ as follows. If e' is an arc of $\overrightarrow{G}$, we set $\overline{f}(\overline{e'}) = f(e')$ for each arc $\overline{e'}$ in the preimage $p^{-1}(e')$. The "lifted" function $\overline{f}$ on arcs of $\tilde{G}'$ defines obviously an n-flow of the graph $\tilde{G}$. $\diamond$

In particular, if G has a 4-flow, then the covering graph $\tilde{G}$ also has a 4-flow. Moreover, if G is an uncolorable cubic graph, then $\tilde{G}$ is also so. A similar statement holds for multigraphs.

Question 1. What are the conditions under which the covering graph of an uncolorable graph is an uncolorable graph?

The class of uncolorable cubic graphs G obtained via covering maps of simple graphs and multigraphs of degree 3 includes the well known subclasses of them such as Isaac's flowers, Goldberg snarks *etc*. Below we describe a general concept of these coverings.

Let G be a connected cubic (multi)graph that is a 4-snark and let $p{:}\,\tilde{G} \to G$ be an n-fold covering of connected graphs that is defined via a permutation voltage assignment $\lambda{:}\,E(\overrightarrow{G}) \to \Sigma_n$. Moreover let $E' = \{e_1, e_2, \ldots, e_r\}$ be a set of arcs in $\overrightarrow{G}$ (here we use the same notations e_i for arcs in $\overrightarrow{G}$ and corresponding edges in G). Cutting the edges $e_1, \ldots, e_r$ in interior points, we shall obtain a $2r$-pole L' with the r "initial" semiedges $e_1', e_2', \ldots, e_r'$ and the corresponding r "terminal" semiedges, denoted by $e_1'', e_2'', \ldots, e_r''$. Assume that E' satisfies the following conditions:

i) the multipole L' is connected;

ii) for each oriented cycle c in $G - E'$ we have $\lambda(c) = e$ where e is the identity permutation of Σ_n.

Let $\lambda(f)(j)$ denote the value of the permutation $\lambda(f)$ at the number j. Now take the initial semiedge e_1' in L'. Since L' is connected, for each terminal semiedge e_m'' where $m = 1, \ldots, r$, there is an oriented path w_m joining e_1' to e_m''. Similarly, for each $h = 1, \ldots, r$ there is in L' a path u_h joining e_1' to e_h' (if $h = 1$, then the path u is trivial). For each $i = 1, \ldots, n$ put $\lambda_m''(i) = \lambda(w_m)(i)$. Here in $\lambda(w_m)$ we count the value of λ at the oriented edges and the output semiedge of the path w_m. Moreover for each $h = 1, 2, \ldots, r$ put $\lambda_h'(i) = \lambda(u_h)(i)$. Here in $\lambda(u_h)$ we count the value of λ at the oriented edges of the path u_h only. In particular, $\lambda_1'(i) = 1$. By condition ii), the permutation λ_m'' does not depend on the choice of the path w_m. Similarly, the permutation λ_h' does not depend on the choice of the path u_h. It follows that the numbers $\lambda_m''(i)$ and $\lambda_h'(i)$ are well defined for each $i = 1, \ldots, n$ and $m = 1, \ldots, r$ and $h = 1, \ldots, r$. In particular, we have $\lambda_1'(i) = i$.

We shall say that the set E has the property iii) if for each $i = 1, \ldots, n$ and each $m = 1, \ldots, r$ it holds the following: $\lambda_m''(i) = \lambda_m'(\lambda_1''(i))$ (see Fig. 3).

We can associate with L' a transition digraph D' as follows. If L' is uncolorable we put formally $D' = \emptyset$. Assume that L' is colorable. An r-tuple $v = (x_1, \ldots, x_r) \in (\mathbf{Z}_2 \times \mathbf{Z}_2)^r$ is an vertex of D' if and only if there exists 4-flow on L' with the input data $(x_1, \ldots, x_r)$ at the sequence of initial semiedges $e_1', \ldots, e_r'$ or this output data at the sequence of terminal semiedges $e_1'', \ldots, e_r''$ of L'. There is an arc in D' that joins the vertex $v = (x_1, \ldots, x_r)$ to the vertex $w = (y_1, \ldots, y_r)$ if and

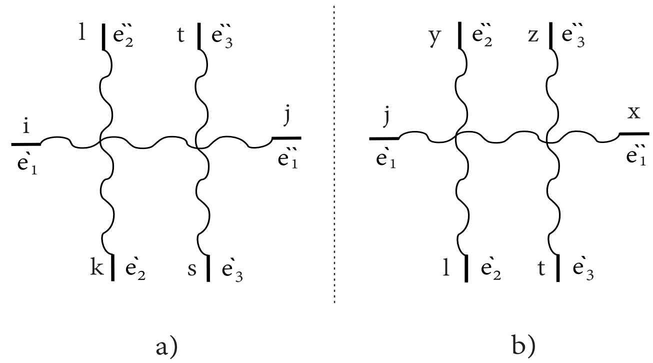

Figure 3: Condition iii) for $r = 3$; here $\lambda_1''(i) = j, \lambda_2''(i) = l, \lambda_3''(i) = t, \lambda_2'(j) = l, \lambda_3'(j) = t$

only if there is a 4-flow on L' with the input data $(x_1, \ldots, x_r)$ and the output data $(y_1, \ldots, y_r)$. Note that since G is assumed to be uncolorable, the digraph D' is loopless.

Theorem 2.2 *Assume that there exists a set E of edges in G which satisfy the above conditions i) - iii). Then G^λ is colorable if and only if there is in D' a closed oriented walk of length n.*

Proof. Let E' be the set of arcs in $\overrightarrow{G}$ with the properties under assumption. Cutting in G all edges e from E', we shall obtain a multipole L'.

By the conditions i) and ii), the multipole $p^{-1}(L')$ is decomposed into n disjoint (isomorphic) copies $L_1, \ldots, L_n$ of the multipole L'. It may occur that L' is not 3-edge colorable i.e. does not have a 4-flow. It follows immediately that $p^{-1}(L')$ is not 3-edge colorable, so the graph G^λ is also uncolorable. Assume now that L' is 3-edge colorable.

Let $(e_i', 1), \ldots, (e_i', n)$ be the lifts of the semiedge e_i' under the covering map p, where $i = 1, \ldots, r$. Similarly, for each $i = 1, \ldots, r$ let $(e_i'', 1), \ldots, (e_i'', n)$ be the lifts of the semiedge e_i'' under the covering map p. By iii), the covering graph G^λ can be obtained from multipoles $L_1, \ldots, L_n$ by identifying consequently the corresponding pairs of their semiedges. More precisely, we identify the semiedges (e_m'', a) and (e_m', b) of the corresponding multipoles L_x and L_y if $\lambda(e_m)(a) = b$. The condition iii) guarantes us that if some terminal semiedge (e_m'', a) of the copy L_x is glued to an initial semiedge (e_m', b) of the copy P_y, then any other terminal semiedge of L_x is identified with some initial semiedge of the copy L_y. This means that after the identification process, the joined multipoles L_x in G^λ are decomposed into cycles. But, by the assumption, the graph G^λ is connected, so we have actually one such a cycle. It follows that $p: G^\lambda \to G$ is a kind of "cyclic" coverings of (multi)graphs.

The existence in D' a closed (oriented) walk w of length n means that one can find 4-flows $\varphi_1, \varphi_2, \ldots, \varphi_n$ on the multipoles $L_1, L_2, \ldots, L_n$, respectively, that are consistent on the output/input data for each pair L_x and L_y of multipoles such that the terminal semiedges of L_x are identified with the corresponding initial semiedges of L_y. Combining all φ_i in the total cyclic sequence, we shall obtain a 4-flow φ of the covering graph G^λ.

Since the multipoles L_x are glued to each other in G^λ cyclically (with a cycle of length n), the inverse implication is rather obvious.$\diamond$

Theorem 2.2 describes cyclic coverings of cubic graphs which allow to obtain a wide class of snarks (which includes Isaac's flowers, Goldberg snarks and many other uncolorable graphs).

Now we consider some coverings of cyclically 4-edge connected and cyclically 6-edge connected uncolorable graphs which do not overlap by Theorem 2.2.

Example 3. Let G be an uncolored cubic graph and let e, f be the non incident edges of G. Cutting the edges e and f in G, we shall obtain a 4-pole L with two pairs of semiedges, e' and e'', and f' and f'', respectively. Then either L does not have any 4-flow or L admits a nowhere-zero flow ξ in $\mathbf{Z}_2 \times \mathbf{Z}_2$ with the following property (*):

ξ has the only nontrivial sources at four semiedges, i.e. $\xi(e') = x, \xi(e'') = y$ and $\xi(f') = x, \xi(f'') = y$ or $\xi(e') = x, \xi(e'') = y$ and $\xi(f'') = x, \xi(f') = y$ where $x, y \in \mathbf{Z}_2 \times \mathbf{Z}_2, x, y \neq 0$ and $x \neq y$.

Take an orientation of the edges of the graph G and denote the resulting orgraph by $\overrightarrow{G}$. Let $\beta\colon \overrightarrow{G} \to \Sigma_5$ be a permutation voltage assignment defined in the following way: $\beta(e) = (12345)$ and $\beta(f) = (13524)$ and $\beta(h) = (1)(2)(3)(4)(5)$ for any other arc h of the $\overrightarrow{G}$. The voltage graph (G, β) defines the 5-fold covering map $p\colon G^\beta \to G$ in which the covering graph G^β is connected.

Proposition 2.3 *The covering $p\colon G^\beta \to G$ is regular and the cubic graph G^β is uncolorable. Moreover if G is cyclically 4-edge connected, and e and f lie on some disjoint circles of G, then G^β is also cyclically 4-edge connected.*

Proof. First note that the set of arcs $E' = \{e, f\}$ satisfies the conditions i) and ii) of Theorem 2.2. It follows that the 20-multipole $p^{-1}(L)$ is decomposed into 5 disjoint copies L_i of the the 4-pole L.

Let $e_1, \ldots, e_5$ be the lifts of the edge e and $f_1, \ldots, f_5$ the lifts of the edge f via the covering map p. Moreover let $e'_1, \ldots, e'_5$ and $e''_1, \ldots, e''_5$ be the lifts of semiedges e' and e'', respectively, and $f'_1, \ldots, f'_5$ and $f''_1, \ldots, f''_5$ be the lifts of semiedges f' and f'', respectively. The covering graph G^β can be obtained in the following way. Take the five copies $L_1, L_2, \ldots, L_5$ of the multipole L. Then identify the 5 pairs of semiedges e'_i and e''_j according to the permutation $\beta(e) = (12345)$ and the 5 pairs of semiedges f'_k and f''_t according to the permutation $\beta(f) = (13524)$ (see Fig. 4). Identifying the first five pairs of semiedges results in the edges $e_1, e_2, \ldots, e_5$ and the second five pairs of semiedges results in the edges $f_1, \ldots, f_5$ of the graph G^β.

The deck transformation group of the covering p is $\mathbf{Z}_5$ which acts on G^β transitively. More precisely, the generator 1 of $\mathbf{Z}_5$ shifts the edge e_i to the edge e_{i+1} and the edge f_j to the edge f_{j+1} for each $i, j = 1, \ldots, 5$. Moreover 1 permutes the components L_i of $p^{-1}(L)$ cyclically. It follows that p is a regular 5-fold covering of connected topological graphs.

If L does not have any 4-flow it follows immediately that G^β is uncolorable. If L admits a nowhere zero $\mathbf{Z}_2 \times \mathbf{Z}_2$-flow, one can directly check that no such flow can be extended to a 4-flow of the covering graph G^β. We omit here the technical details of this checking (which relies actually on the property (*) of the 4-poles $L_i, i = 1, \ldots, 5$).

If the edges e and f of G lie on some disjoint cycles and the permutations $\beta(e)$ and $\beta(f)$ are cyclic, then the covering graph G^β is cyclically 4-edge connected. $\diamond$

Below we provide an example of non regular 6-fold covering map of connected uncolorable cubic graphs.

Example 4. Let G be uncolorable cyclically 4-edge connected cubic graph and let e, f, h be nonincident edges of G. Cut the edges e, f and h of G in internal points. As a result, we obtain a 6-pole M with corresponding pairs of semiedges , e' and e'', f' and f'', h' and h'', respectively. Assume that cutting any two of the edges e, f and h produces an uncolorable 4-pole. Then either 1)

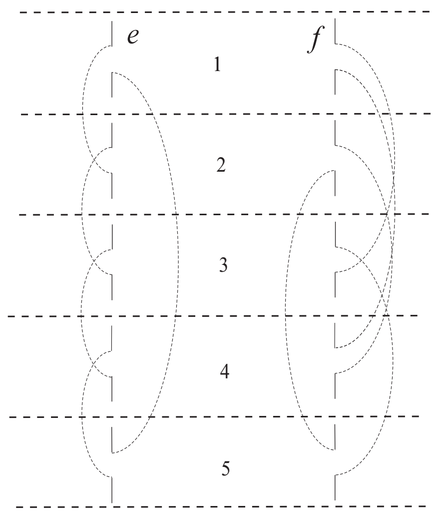

Figure 4: Obtaining the graph G^{β} by gluing the five copies of the 4-pole L

M does not admit 3-edge coloring, or 2) M has a 3-edge coloring ξ with the following combination of (nonzero) colors $x, y, z \in \mathbf{Z}_2 \times \mathbf{Z}_2, x, y, z \neq 0$ at three pairs of semiedges:

(**) $\xi(e') = x, \xi(e'') = y$ and $\xi(f') = y, \xi(f'') = z$ and $\xi(h') = z, \xi(h'') = x$,

or any other combination obtained from the given one by permutation of colors in corresponding pairs of semiedges. Note that, in the case 2), the distribution of colors such that the pair of semiedges e and e' (or any other pair of two corresponding semiedges) get the same color is not admissible since $r(G) \geq 3$. For example, the following distribution of colors is admissible: $\xi(e') = x, \xi(e'') = y$ and $\xi(f'') = z, \xi(f') = y$ and $\xi(h') = z, \xi(h'') = x$.

Consider a 5-fold covering of the graph G given by the permutation voltage assignment φ with values in the symmetric group Σ_5 on the orgraph G' as follows (see Fig. 5).

Put $\varphi(e) = (12345), \varphi(f) = (153)(24), \varphi(h) = (142)(35)$. For the remaining edges g put $\varphi(g) = (1)(2)(3)(4)(5)$.

Proposition 2.4 *The covering $p\colon G^{\varphi} \to G$ is nonregular and the cubic graph G^{φ} is uncolorable. Moreover if the graph G is a cyclically 6-edge connected and the edges e, f and h are contained in three disjoint circles of G, then G^{φ} is also uncolorable cyclically 6-edge connected cubic graph.*

Proof. Note that the set of edges $E' = \{e, f, h\}$ satisfies the conditions i) and ii) of Theorem 2.2. It follows that the multipole $p^{-1}(M)$ is decomposed into 5 disjoint copies M_i, isomorphic to the 6-pole M.

Let $e_1, \ldots, e_5$ be the lifts of the edge e, $f_1, \ldots, f_5$ be the lifts of the edge f and $h_1, \ldots, h_5$ be the lifts of the edge h via the covering map p. Moreover let $e'_1, \ldots, e'_5$ and $e''_1, \ldots, e''_5$ be the lifts of semiedges e' and e'', respectively, $f'_1, \ldots, f'_5$ and $f''_1, \ldots, f''_5$ be the lifts of semiedges f' and f'', and $h'_1, \ldots, h'_5$ and $h''_1, \ldots, h''_5$ be the lifts of semiedges h' and h'', respectively. The covering graph G^{φ} can be obtained in the following way. Take the five copies $M_1, M_2, \ldots, M_5$ of the multipole M. Then identify the 5 pairs of semiedges e'_i and e''_j according to the permutation $\varphi(e) = (12345)$,

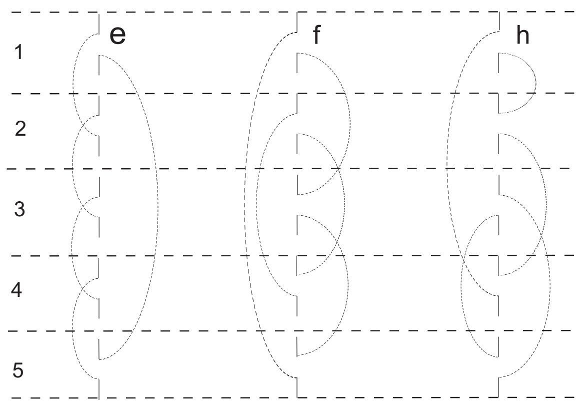

Figure 5: Obtaining the graph G^{φ} by gluing the five copies of the 6-pole M

the 5 pairs of semiedges f'_k and f''_t according to the permutation $\varphi(f) = (153)(24)$ and the 5 pairs of semiedges h'_k and h''_t according to the permutation $\varphi(h) = (142)(35)$ (see Fig. 5). Identifying the first five pairs of semiedges results in the edges $e_1, e_2, \ldots, e_5$, the second five pairs of semiedges results in the edges $f_1, \ldots, f_5$ and the third five pairs of semiedges results in the edges $h_1, \ldots, h_5$ of the graph G^{φ}.

The deck transformation group of the covering p is trivial, so this covering is nonregular.

If M does not have any 4-flow it follows immediately that G^{φ} is an uncolorable graph. If M admits a nowhere zero $\mathbf{Z}_2 \times \mathbf{Z}_2$-flow, we can directly check that no such flow can be extended to a 4-flow of the covering graph G^{φ}. The proof of the last fact is by counting all possible subcases of extending a coloring of one copy of the 6-pole M to the whole covering graph and uses the property (**) of multipoles $M_i, i = 1, \ldots, 5$. We omit here the technical details.

If the edges e, f and h of G lie on disjoint cycles and the permutations $\varphi(e)$, $\varphi(f)$ and $\varphi(h)$ are cyclic, the covering graph G^{φ} is cyclically 6-edge-connected. $\diamond$

3 Coverings of cubic graphs and resistance

In [11], Steffen introduced the parameter $r(G)$ of an uncolorable cubic graph G without bridges. It measures how far G is from being 3-edge colorable and is called the resistance of G. More precisely, $r(G) = min\{|F| : F \subset E(G)$ such that $G - F$ is 3-edge colorable$\}$ (here we slightly modify the original definition of the parameter $r(G)$ but in an equivalent form). This parameter is related to another measure of noncolorability, the oddness $\omega(G)$ of G, which is the smallest possible number of odd circuits in 2-factors of G (see [6, 11]). In particular, $r(G) \leq \omega(G)$ for any cubic graph G.

It is not difficult to see that the number $r(G)$ is equal to the minimal number of edges in the cubic graph G, say $e_1, \ldots, e_k$, such that cutting all them in interior points results in a $2k$-pole which has a 4-flow (with sources in the semiedges).

It follows directly from definitions that an analogue of Proposition 2.3 holds true for uncolorable cubic graphs G with $r(G) \geq 3$ and for an arbitrary choice of cyclic permutations $\beta(e)$ and $\beta(f)$ in Σ_n with $n \geq 2$.

Let us consider several examples of snarks and indicate their resistance.

Example 5. Let P be the Petersen graph, and P^3 the third power of P pictured in Fig. 1. In Fig. 6, it is shown the snark $\mathcal{G}_{26}$ of order 26 embedded in a torus (see [10]).

By direct computation, we have $r(P) = 2$, $r(P^3) = 2$ and $r(\mathcal{G}_{26}) = 2$.

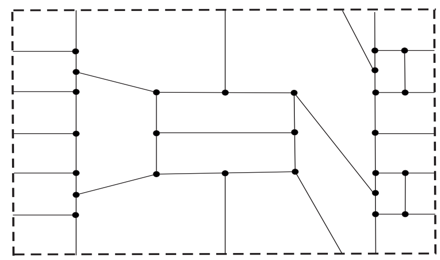

Figure 6: The snark $\mathcal{G}_{26}$ embedded in torus

The following theorem allows to construct uncolored graphs G with an arbitrary value of resistance.

Theorem 3.1 *Let G be a connected bridgeless uncolored graph with $r(G) = k$. Let (G, μ) be a permutation voltage graph with an assignment μ in the symmetric group Σ_n, G^μ be the corresponding covering graph and let $E = \{e_1, e_2, \ldots, e_l\}$ be a subset of edges of G with $l \leq k-1$. Assume that E satisfies the following conditions:*

i) the graph $H = G - E$ is connected;

ii) for each oriented cycle c in the graph H we have $\mu(c) = e$ where e is the trivial permutation in Σ_n.

Then the bridgeless cubic graph G^μ is uncolored. Moreover $r(G^\mu) \geq (k-l)n$.

Proof. First assume that the covering graph G^μ is connected. Let L be the $2l$-pole associated with the set of edges $E = \{e_1, \ldots, e_l\}$ in G, that is L is obtained from G by cutting the edges $e_1, \ldots, e_l$ in interior points. It follows from i) and ii) (see the proof of Theorem 2.2) that the multipole $p^{-1}(L)$ is decomposed into n disjoint (isomorphic) copies L_i of the multipole L. Moreover the covering graph G^μ can be obtained from multipoles $L_1, \ldots, L_n$ by identifying the corresponding pairs of their semiedges in accordance with the permutation values $\mu(e_i)$ for $e_i \in E$. It follows that the graph $p^{-1}(H)$ is decomposed into n disjoint components $H_1, H_2, \ldots, H_n$ each of which is isomorphic to H. It is clear that each graph H_i is obtained from the multipole L_i by removing all semiedges of it.

Suppose that $r(G^\mu) = t < (k-l)n$. Then there are edges $e'_1, \ldots, e'_t$ of G^μ such that the graph $U = G^\mu - \{e'_1, \ldots, e'_t\}$ is 3-edge colorable. Let φ be the corresponding 3-edge coloring of U. Then φ descends obviously to a proper 3-edge coloring φ_i of each subgraph $U_i = H_i - \{e'_1, \ldots, e'_t\}$ where $i = 1, \ldots, n$. Since U_i is 3-edge colorable and $r(G) = k \geq 2$, it follows that the graph U_i is obtained from the graph H_i by removing at least $k-l$ edges, $i = 1, \ldots, n$. This means that the U is obtained from G by eliminating at least $(k-l)n$ edges, i.e. $r(G^\mu) \geq (k-l)n$, contradicting our assumption.

If G^μ is disconnected, we can restrict the covering map $p\colon G^\mu \to G$ to each connected component of G^μ and then argue in the same way as in the first case. Now the assertion follows. $\diamond$

Note that by Proposition 1.1, for any n-fold covering $p\colon G^\mu \to G$ of cubic graphs with $r(G) = k$, we have $r(G^\mu) \leq kn$.

Corollary 3.1 *Let G be a connected bridgeless uncolorable graph with $r(G) = k$ and $\overrightarrow{G}$ be an orientation of G. Moreover let (G, μ) be a voltage graph with a voltage assignment $\mu\colon E(\overrightarrow{G}) \to A$ in a finite group A of order m such that μ takes the only nontrivial values at l arcs of $\overrightarrow{G}$ with $l \leq k-1$. Then the covering cubic graph G^μ is an uncolorable graph with $r(G^\mu) \geq (k-l)m$.*

Proof. Let $E = \{e_1, e_2, \ldots, e_l\}$ be the edges of G with nontrivial values of the voltage assignment μ. If the graph $G - \{e_1, e_2, \ldots, e_l\}$ is connected, the assertion is a direct consequence of Theorem 3.2. If $G - \{e_1, e_2, \ldots, e_l\}$ is disconnected, we can replace the set E with a smaller subset $E' \subset E$ such that $G - E'$ is connected and the condition iii) of Theorem 3.2 is satisfied. Now the assertion follows from the proof of Theorem 3.2. $\diamond$

Example 6. Consider the 5-fold covering of the graph H_2 depicted in Fig. 7. This uncolorable graph is due to [8].

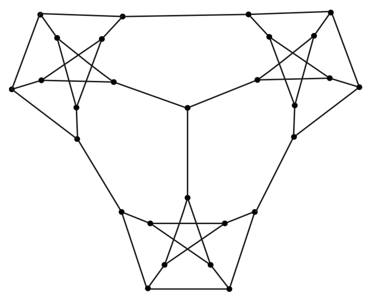

Figure 7: The cubic graph H_2

In [8], it was shown that H_2 is a unique smallest uncolorable graph with oddness 4 and with edge-cyclic connectivity 3. The given graph is obtained by gluing together three copies of the 3-pole P_3 [8], where the multipole P_3 is shown in Fig. 8.

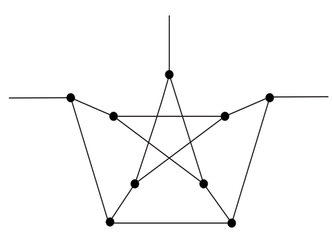

Figure 8: The multipole P_3

The order of H_2 is equal to 28. It is not difficult to show that $r(H_2) = 3$. We distinguish in H_2 three edges, e, f and g, and consider the 5-fold covering map $p\colon H_2^\beta \to H_2$ defined via the permutation voltage assignment β with values in Σ_5 as follows: $\beta(e) = (12345)$, $\beta(f) = (153)(24)$, $\beta(g) = (142)(35)$ and $\beta(h) = (1)(2)(3)(4)(5)$ for any other arc h of the orgraph $\overrightarrow{H_1}$. It is clear that $r(H_2^\beta) \geq 3 \cdot 5$ since in order to obtain from H_2^β an uncolored (subcubic)graph we have to delete at least one edge in each copy $P_3^i, i = 1, 2, \ldots, 15$ of the 3-pole P_3. Since the number $r(H_2^\beta)$ cannot exceed $3 \cdot 5$, it follows that $r(H_2^\beta) = 15$. Note that edge-cyclic connectivity of H_2^β is also 3.

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