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The Optimal General Upper Bound for the Distinguishing Index of Infinite Graphs*

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Abstract

The *distinguishing index* $D'(G)$ of a graph G is the least cardinal number d such that G has a edge-colouring with d colours which is preserved only by the trivial automorphism.

We use a new method to prove a general upper bound $D'(G) \leq \Delta - 1$ for any connected infinite graph G with finite maximum degree Δ that is not a double ray. This is in contrast with finite graphs since for every $\Delta \geq 3$ there exist infinitely many connected, finite graphs G with $D'(G) = \Delta$. We also give examples showing that this bound is sharp for any maximum degree Δ.

Keywords: edge colouring; symmetry breaking in graph; distinguishing index; infinite graph; automorphism.
Mathematics Subject Classifications: 05C15, 05C25, 05C63

1 Introduction

We say that an automorphism φ of a graph G preserves an edge-colouring $c : E(G) \to C$ if $c(xy) = c(\varphi(x)\varphi(y))$ for every $xy \in E(G)$. If c is not preserved by an automorphism φ we say that c *breaks* φ. The least cardinal

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number d such that there exists an edge-colouring c with d colours breaking all nontrivial automorphisms of G is called the *distinguishing index* of G and is denoted by $D'(G)$. It is well defined for every connected graph which is not isomorphic to a path of length one. The definition of $D'(G)$ was introduced in [3] by Kalinowski and Pilśniak and it is similar to the notion of the *distinguishing number* $D(G)$ defined for vertex colourings by Albertson and Collins in [1].

Assume that the graph G has a (partial) edge-colouring c. We say that a vertex v is *fixed*, if it is fixed by every automorphism of G that preserves colouring c. Similarly, we say that the set $A \subset V(G)$ is *fixed* if it is fixed pointwise by every automorphism of G that preserves colouring c.

Kalinowski and Pilśniak proved the following upper bound for distinguishing index of finite graphs.

Theorem 1 [3] *Let Δ be any cardinal number. If G is a connected, finite graph of order $n \geq 3$, then $D'(G) \leq \Delta(G)$ unless $G = C_3,\ C_4$ or C_5.*

This concept was also investigated for infinite graphs. Broere and Pilśniak obtained the following bound for infinite graphs similar to the one given above.

Theorem 2 [2] *Let G be a connected, infinite graph such that the degree of every vertex is not greater than Δ. Then $D'(G) \leq \Delta$.*

The aim of this paper is to improve this result and to show that the bound given in Theorem 3 is best possible for every finite $\Delta \geq 3$.

Theorem 3 *Let G be a connected, infinite graph with finite maximum degree $\Delta \geq 3$. Then $D'(G) \leq \Delta - 1$.*

We prove this Theorem separately for graphs with maximum degree three in Section 3, and for other graphs in Section 2.

2 Bound for graphs with $\Delta(G) \geq 4$

A *ray* is a graph $G = (V, E)$, whose vertices may be enumerated with positive integers, i.e. $V = \{x_1, x_2, \dots\}$ such that two vertices are adjacent if and only if they are enumerated with consecutive numbers. a *double ray* is a graph formed by gluing two rays in vertices of degree one. We say that a graph is *locally finite* if the degree of every vertex is finite. Before we prove the Theorem 3 for infinite graphs G with $\Delta \geq 4$, we formulate and prove the following lemma.

Lemma 4 *Let G be a connected infinite, locally finite graph. Then there exists a non-empty maximal subgraph of G whose every component is a ray.*

Proof. Let G be a connected, infinite locally finite graph. Let $\mathcal{S}$ be a family of subgraphs of G whose every component is a ray or a double ray. By Kőnig's Lemma it is non-empty. Elements of $\mathcal{S}$ are ordered by the subgraph relation and let $\mathcal{C}$ be a non-empty chain in $\mathcal{S}$. We show that the union $\bigcup \mathcal{C} = \left(\bigcup_{C \in \mathcal{C}} V(C), \bigcup_{C \in \mathcal{C}} E(C) \right)$ is an upper bound of $\mathcal{C}$. Suppose that this is not true. Then there exists a component A of $\bigcup \mathcal{C}$ which is neither a ray nor a double ray. It means that it is constructed as the sum of components (rays, double rays) of elements of $\mathcal{C}$. It follows that there exist some $B_1, B_2 \in \mathcal{C}$ and its components $C_1 \subset B_1$, $C_2 \subset B_2$, $C_1 \neq C_2$ such that $C_1 \cup C_2$ is not a ray nor a double ray. As $\mathcal{C}$ is a chain then either $B_1 \subset B_2$ or $B_2 \subset B_1$ and $C_1 \cup C_2$ is a subgraph of B_1 or B_2 but this is a contradiction because all components are (double) rays and $C_1 \cup C_2$ cannot be a subgraph of any (double) ray. Therefore by Zorn's Lemma there exists a maximal subgraph of G whose every component is a ray or double ray. We obtain a subgraph that satisfies the claim by deleting one edge from every double ray. $\square$

Now, we formulate some additional definitions and a useful theorem. A *symmetric tree* is a finite tree with a central vertex v_0 (i.e. fixed by every automorphism), all leaves are at the same distance from v_0 and all vertices who are not leaves have the same degree. A *bisymmetric tree* is a finite tree with a central edge v_0 (i.e. fixed by every automorphism), all leaves are at the same distance from e_0 and all vertices who are not leaves have the same degree. The following theorem was proved by Pilśniak.

Theorem 5 [4] *If G is a connected, finite graph of maximum degree $\Delta(G)$ at least three, then $D'(G) \leq \Delta(G) - 1$ unless G is a symmetric tree, a bisymmetric tree, $K_{3,3}$ or K_4, when $D'(G) = \Delta(G)$.*

We can now prove the bound for infinite graphs G with finite maximum degree $\Delta(G) \geq 4$.

Theorem 6 *If G is a connected infinite graph with finite maximum degree $\Delta \geq 4$, then $D'(G) \leq \Delta - 1$.*

Proof. Let F be a maximal subgraph of G whose every component is a ray. Denote by $\mathcal{R} = \{R_i : i = 1, 2 \ldots, \alpha\}$, for some $\alpha \in \{1, 2, \ldots, \omega\}$, the set of components of A. We now define a partition of $E(G)$ onto sets A, B, C and D as follows:

$$A = E(F),$$
$$B = \{e \in E(G) : |e \cap V(F)| = 2\} \backslash A,$$
$$C = \{e \in E(G) : |e \cap V(F)| = 1\},$$
$$D = E(G) \backslash (A \cup B \cup C).$$

We want to show that there exists a colouring $c : E(G) \to \{1, \ldots, \Delta - 1\}$ breaking all non-trivial automorphisms. First, we fix vertices of $V(F)$. We colour A blue and red. We colour each ray R_i from $\mathcal{R}$ by coding a repeating sequence, unique to each ray. The word (blue, blue, red) corresponds to number 1 and the word (blue, red) corresponds to number 0. Denote by w_i the word $10 \ldots 0 = 10^i$, where 0 occurs i times. We code a repeating sequence $w_i^\omega := w_i w_i w_i \ldots$ by colouring edges of R_i with colours blue and red. We colour each edge of B with yellow colour (as $\Delta - 1 \geq 3$, we have at least three colours).

The graph $G - V(F)$ has only finite components because of the maximality of F. Each of these components is joined by an edge from C to at least one of the rays in $\mathcal{R}$. We define an equivalence relation on components of $G - V(F)$. We say that two components of $G - V(F)$ are in relation if they have the same neighbours in $V(F)$. An endvertex of any ray from $\mathcal{R}$ is not incident with any edge in C. Otherwise, we could extend that ray by that edge to obtain a greater subgraph F' whose every component is a ray which is a contradiction with the maximality of F. It follows that every vertex in $V(F)$ is incident with no more than $\Delta - 2$ edges from C. For every equivalence class of the defined relation we choose and mark one vertex of F in such a way that every component of $G - V(F)$ is connected to exactly one marked vertex. For each marked vertex we colour incident edges in C with different colours without using red or without using blue. If u_i is a marked vertex in a ray $R = \{u_1, \ldots, u_{i-1}, u_i, \ldots\} \in \mathcal{R}$ and the edge $u_{i-1}u_i$ is blue, then we do not use blue for any edge of C incident with u_i. Similarly, we do not use red if the edge $u_{i-1}u_i$ is coloured with red. The edges in C not incident with any marked vertex we colour with yellow. Moreover, we do not use yellow for edges connecting a component H with a marked vertex if there are more than one edge from H to a marked vertex. Hence, if the vertices of $V(F)$ are fixed, then for every component H there exists a fixed edge incident with some marked vertex. The edges in D will be coloured later.

Now we want to show that by this (partial) colouring we fixed every ray in $\mathcal{R}$ no matter how the remaining part of graph is coloured. Notice that the only rays coloured entirely with blue and red are rays in $\mathcal{R}$, and rays which are obtained by joining a path contained in one of the components of $G - A$, and infinite subrays of some ray in $\mathcal{R}$. This is because every component of $G - A$ is connected to $V(F)$ only by one blue or red edge, so a ray containing only blue and red edges that starts in $V(F)$ and containing at least one vertex of $G - V(F)$ does not exist. Every ray $R \in \mathcal{R}$ has to be mapped onto a ray which contains an infinite subray of R due to the periodic coding. Assume that R_i (for some i) may be mapped onto a distinct ray and denote it by $R' = e_0 e_1 \dots e_n \cup R''$, where R'' is a subray of R_i and $e_n \in C$. If the edge e_n is coloured with red, then R' contains two adjacent red edges but R does not, or in the coding of R' there exists a word $10^j 1$ for some $j < i$. Suppose that e_n is coloured with blue. In that case R' contains a blue path of length three which does not occur in R, or in the coding of R' there exists a word $10^j 1$ for some $j < i$. We thus showed that each vertex in $V(F)$ is fixed.

Now we colour the edges in D. Each vertex from $V(F)$ is fixed, and the components of $G - V(F)$ connected to the same set of vertices in $V(F)$ are distinguished by colours of the edges incident with marked vertices. Therefore, each component of $G - V(F)$ has to be mapped onto itself. It remains to show that we can break all non-trivial automorphism of every component H of $G - V(F)$ with $\Delta - 1$ colours. Denote by H' the graph $H + e$, where e is an already fixed (pointwise) edge incident with H and with a marked vertex. If H is P_2 or a cycle, then we can distinguish H' with two colours leaving the colour on e as it was before. If H is a bisymmetric tree or a symmetric tree of degree Δ, then H' is not such a tree anymore. If H is equal to $K_{3,3}$ or K_4, then $\Delta(H) < \Delta(G)$. Therefore we can distinguish H or H' (colouring edge e as it was coloured before) with $\Delta - 1$ colours by Theorem 5. We thus constructed a colouring that fixes every vertex of G. $\square$

3 Bound for graphs with $\Delta(G) = 3$

We need one more definition and theorem proved by Broere and Pilśniak in [2]. Let x be a vertex in G. Denote by $B_x(k)$ the closed ball of radius k and with center x as the set of vertices of distance from the vertex x not greater than k. We say that a connected graph G is a *tree-like graph* if it contains a vertex x with the property that for any vertex $y \neq x$ there exists a vertex z such that $\{y\} = B_z(1) \cap B_x(d(x,z) - 1)$. Notice that an infinite tree T is a tree-like graph if and only if it has at most one leaf.

Theorem 7 [2] *If G is a tree-like graph such that the degree of every vertex is not greater than $2^{\aleph_0}$, then $D'(G) \leq 2$.*

We can now prove the remaining part of Theorem 3.

Theorem 8 *Let G be a connected infinite graph with maximum degree $\Delta(G) = 3$. Then $D'(G) \leq 2$.*

Proof. Let x be a vertex of degree three. Let T be a spanning BFS tree of G rooted at x. We say that a vertex v of G has a *standard colouring* if the edges joining v with its sons in T have distinct colours. Denote by t_1, t_2 and t_3 the neighbours of x. We define a colouring of the edges of T. The edges of $G - T$ we colour $G - T$ with red. Roughly speaking, to show that the colouring of G is distinguishing, it is enough to show that x is fixed and all vertices whose fathers are not coloured with standard colouring are fixed.

Observe that if we have a partial colouring of G such that the fathers of u and v are fixed in G and they have a standard colouring, then u cannot be mapped to v by any automorphism which preserves the colouring. Suppose this is not true, and u can be mapped to v for some u, v with fixed fathers. If u or v is connected to its father with a blue edge, then this vertex is fixed, so assume that both v and u are connected to their fathers with red edges. This means that u is connected in G with a red edge with father of v and v is connected in G with a red edge with father of u, so u and v have the same father in T because it is a BFS tree. This contradicts the standard colouring of the father of u. It follows that to show that the colouring of G is distinguishing, it is enough to show that the root x is fixed in G and all vertices whose fathers have two edges connecting them with their sons with same colour are fixed.

For $i = 1, 2, 3$, let T_i be the component of $T - x$ containing t_i. Now, we consider three cases.

Case 1. At least two of T_1, T_2 and T_3 are infinite and one of them, say T_1, has no vertices of degree one in T, and have at most finitely many vertices of degree two in T.

If every vertex of T has degree three in T, then $G = T$ is a tree-like graph and there exists a distinguishing colouring of G with two colours by Theorem 7. We may assume that T_3 is infinite and T_2 has a vertex of degree less than three in T. We colour each edge incident with x with blue. We can colour T_2 with a standard colouring such that there is no blue ray in $T_2 + x$ with endvertex x. We colour T_3 with a standard colouring such that there exists a blue ray with endvertex x in $T_3 + x$. We colour one ray in $T_1 + x$ with endvertex x with blue, and all the edges incident with any vertex in this ray with blue (there are infinitely many such edges). The remaining part of T_1 is coloured with a standard colouring. If we colour $G - T$ with red, then the vertex x is fixed in G because it is a unique vertex v in G such that in all double rays R containing v, all vertices with three incident blue edges are in the same component of $R - v$. Every vertex in T connected with x with a blue path is fixed and the remaining vertices are fixed because of the standard colouring.

Case 2. At least two of the trees T_1, T_2 and T_3, say T_1 and T_3, are infinite and all infinite trees among T_1, T_2 and T_3 have either a vertex of degree one in T or infinitely many vertices of degree two in T.

We colour each edge incident with x with blue. As T_2 has a vertex of degree less than three in T, we can colour $T_2 + x$ with a version of standard colouring in which there is no blue ray with endvertex x. Let P be a maximal blue path in $T_2 + x$ with endvertex x. As T_1 is infinite we can colour $T_1 + x$ with a version of standard colouring in which there exists a blue ray with endvertex x.

Assume first that there exists a vertex $y \in T_3$ such that $d_T(y) = 1$. If the distance between x and y in T is not equal to the length of the path P, then we colour the path from x to y with blue and we colour the remaining vertices so that $T_3 + x$ is coloured with the standard colouring. The vertex x is fixed in G because it is the only vertex incident with three blue edges, t_1 is distinguished from t_2 and t_3 because of the blue ray, and t_2 are distinguished from t_3 because of distinct lengths of maximal blue paths in $T_2 + x$ and $T_3 + x$ with endvertex x. If the distance between x and y in T is equal to the length of P, then we colour the path from x to y with blue and one ray in $T_3 + x$ with endvertex x with blue. The remaining part of T_3 is coloured with the standard colouring. In this case, there are two vertices with three incident blue edges. One of them is x, and let z be the other one. The vertices x and z are fixed in G because the length of a path from z to y is less than the length of P. The vertex t_2 is fixed because there is no blue ray in T_2 and t_3 is contained in a blue ray with endvertex x containing z, while t_1 is not, the remaining part of T is fixed because of the standard colouring.

Finally, assume that there are infinitely many vertices in T_3 with degree two in T. Take one on them, say y, such that the level of y is greater than the length of P. We colour $T_3 + x$ with a version of the standard colouring such that the path connecting x and y is coloured with blue and the edge between y and its son with red. The vertices x, t_1, t_2 and t_3 are fixed with the same reasoning as before.

Case 3. Two of the trees T_1, T_2 and T_3, say T_1 and T_2, are finite.

We colour xt_1 with blue, xt_2 with red and xt_3 with blue. We put $y := t_3$. If $d_T(y) = 2$, let s be its son. Then we colour edges incident with y with blue, we put $y := s$ and we repeat the procedure from checking the degree of y. If $d_T(y) = 3$, let s_1 and s_2 be its sons. Without loss of generality, assume that s_2 is in an infinite component of the tree $T - y$. Let H_1 be the component of $T - y$ containing s_1. If H_1 is finite then we colour the edge ys_1 with red, we put $y := s_2$ and we restart the procedure from checking the degree of y. If H_1 is infinite then we proceed as in Case 1 or 2, by taking putting T_2 equal to the component of $T - y$ containing x, putting $x := y$. We can follow the procedure descibed in both cases and still retaining the colouring on edges already coloured by procedure from Case 3. This colouring fixes the root x, and hence all other vertices. If the procedure never stops, i.e. when all H_1 are finite, then the procedure generates a blue ray. We colour the remaining part of T with the standard colouring. Notice that there exists only one maximal blue ray in T, so each its vertex is fixed in G, and hence x is fixed in G. The remaining vertices are fixed because of the standard colouring. □

4 Sharpness of the bound

From Theorem 6 and Theorem 8 it follows that Theorem 3 holds. We now construct a family of infinite graphs G_k with maximum degree k for

$k = 3, 4, \ldots$, such that $D'(G_k) = \Delta(G_k) - 1 = k - 1$ showing that the bound in Theorem 3 is sharp. Let R be a ray with an endvertex y, and let T_k be a symmetric tree with a central vertex x and with $\Delta(T) = k - 1$. Assume that the sets of vertices of R and T_k are disjoint. We now define the graph $G_k = (V, E)$ so that $V = V(R) \cup V(T_k)$ and $E = E(R) \cup E(T_k) \cup \{xy\}$. The graph G_k has only one vertex x with maximum degree k. The vertices in R and the vertex x are fixed by every automorphism of G_k, so to colour G_k distinguishingly it is sufficient and necessary to colour edges in T_k distinguishingly and color the rest of the graph arbitrarily, so $D'(G_k) = D'(T_k) = k - 1$, where the last equality follows from Theorem 5. This proves that the bound in Theorem 3 is sharp for any finite $\Delta(G) \geq 3$.

References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1996), #R18.

[2] I. Broere and M. Pilśniak, The Distinguishing Index of the Infinite Graphs, Electron. J. of Combin. 23(1) (2015), #P1.78.

[3] R. Kalinowski and M. Pilśniak, Distinguishing graphs by edge colourings, European J. Combin. 45 (2015), 124–131.

[4] M. Pilśniak, Improving Upper Bounds for the Distinguishing Index, Ars Math. Contemp. 13(2017) 259-274.