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Small dense on-line arbitrarily partitionable graphs

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Abstract

A graph $G = (V, E)$ is *arbitrarily partitionable* if for any sequence $(n_1, \ldots, n_k)$ that satisfies $n_1 + \cdots + n_k = |G|$ it is possible to divide V into disjoint subsets $V = V_1 \cup \cdots \cup V_k$ such that $|V_i| = n_i, i = 1, \ldots, k$ and the subgraphs induced by all V_i are connected. In this paper we inspect an on-line version of this concept and show that for graphs of order $n \in \{8, \ldots, 14\}$ and size greater than $\binom{n-3}{2} + 6$ these two concepts are equivalent. Our result together with a theorem of Kalinowski imply that the equivalence between those two concepts holds for graphs of any order n and size greater than $\binom{n-3}{2} + 6$.

Keywords: partitions of graphs, traceable graphs, perfect matching.
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1. Introduction

Let $G = (V, E)$ be a graph of order n. We say that a sequence of positive integers $(n_1, \ldots, n_k)$ is *admissible* if $n_1 + \cdots + n_k = n$. An admissible sequence is *realizable in G* if there exists a partition $V = V_1 \cup \cdots \cup V_k$ such that $|V_i| = n_i$ and $G[V_i]$ is connected, for all $i = 1, \ldots, k$. Then we say that G is *arbitrarily partitionable* (AP, for short) if any admissible sequence is realizable in G. This concept has been introduced by Barth, Baudon and Puech [1] and independently by Horňák and Woźniak [3].

Lately this definition was modified by Horňák, Tuza and Woźniak [2] to establish a new concept, namely on-line arbitrarily partitionable graphs. In this variation we do not have a whole sequence at the start, but we are given the elements individually, in succession. At the i-th step we make a choice of a connected subgraph V_i. If we can finish that scheme for any admissible sequence, we call a graph *on-line arbitrarily partitionable.*

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The problem of finding sufficient conditions for a graph to be (on-line) AP has been intensively studied in last years. In this paper we show that the equivalence between AP and on-line AP condition holds if the order of a graph is between 8 and 14 and if the graph has sufficiently many edges. Thus we fill the gap in a theorem of Kalinowski [4] who has proved that equivalence for all graphs of order at least 15 or at most 7. We state here the main theorem and prove it in section 3.

Theorem 1. *Let G be a connected graph of order $n \in \{8, \ldots, 14\}$ and of size*

$$||G|| > \binom{n-3}{2} + 6. \tag{1}$$

Then G is AP if and only if G is on-line AP.

2. Preliminaries

In the beginning, we prove a few sufficient conditions for a graph satisfying inequality (1) to be traceable.

Lemma 2. *Any graph created from K_n by deleting at most $n-4$ edges is Hamiltonian-connected.*

Proof. Let G be such a graph. Checking Ore's condition for a graph to be Hamiltonian-connected, for any u, v where $uv \notin E(G)$, we get

$$d(u) + d(v) \geq 2(n-1) - (n-4) - 1 = n+1$$

The inequality holds, because there can be only $n-4$ edges missing and only one of them might count twice. □

Lemma 3. *Let G be a connected AP graph of even order that satisfies inequality (1). If there exist two nonadjacent vertices x, y of G such that $d(x) + d(y) \leq 4$, then G is traceable.*

Proof. Assume that there are such vertices x, y. We count the edges of a graph $G' = G - \{x, y\}$.

$$||G'|| = ||G|| - d(x) - d(y) \geq \binom{n-3}{2} + 7 - 4$$

Then, since $|G'| = n - 2$

$$\binom{n-2}{2} - ||G'|| \leq n - 6 = (n-2) - 4$$

There are only $|G'| - 4$ edges missing from K_{n-2}, so by the previous lemma G' is Hamiltonian-connected. Now, either x and y have both only one, the same neighbour (this is a contradiction with G being AP since there is no perfect matching) or they have two different neighbours (then we can extend a Hamiltonian path). □

Now we strenghten these lemmas, using the results of Kewen, Lai and Zhou [7]. First, we explain their notation. Let G and H be two graphs, then $G \cup H$ is a disjoint union of G and H, $G \vee H$ is a graph obtained from G and H by joining every vertex of G with every vertex of H. By G_n we denote any graph of order n. Finally define $G_2 : G_n$ to be any 2-connected graph obtained from $G_2 \cup G_n$ by joining every vertex of G_2 to some vertices of H.

Lemma 4. ([7]) *If G is a graph of order n satisfying $d(x)+d(y) \geq n$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian-connected or $G \in \{G_2 : (K_s \cup K_h), G_{n/2} \vee \overline{K}_{n/2}\}$.*

Basing on this, we prove the following.

Lemma 5. *Any graph G created from K_n ($n \geq 7$) by deleting at most $n-3$ edges is either Hamiltonian-connected or there are Hamiltonian x, y-paths for all $x, y \in V(G)$ except for one pair $\{x_0, y_0\}$.*

Proof. Assume that G is not Hamiltonian-connected. Then by the previous lemma, G belongs to one of two mentioned classes. Graphs in the second class have an independent set of order $\frac{n}{2}$ so they have at least $\binom{n/2}{2}$ edges missing and this number is greater than $n-3$ for $n \geq 7$.

Then the graph G is in the first class, i.e., $G = G_2 : (K_s \cup K_h)$ for some s and h satisfying $s + h + 2 = n$. This graph has size at most

$$1 + 2(s+h) + \binom{s}{2} + \binom{h}{2} = \frac{s^2 + h^2 + 3s + 3h + 2}{2} = \frac{n(n-1)}{2} - hs.$$

Now, $hs = h(n-2-h)$ is not greater than $n-3$ only for $h = 1$ or $s = 1$, which means that the vertex v in K_1 has only 2 neighbours out of $n-1$ possible, so all the excluded edges were incident to v.

To complete the proof we show that the exceptional pair $\{x_0, y_0\}$ consists of the neighbours x, y of v. It is clear that there is no Hamiltonian path between them. Observe that a graph constructed by contracting the edge xv is a complete graph. For two vertices u_1, u_2 different than v we can contract the edge xv, find any u_1, u_2-path containing the edge xy and then expand xv. If $u_1 = v$ and $u_2 \neq x$ we proceed similarly as before, finding x, u_2-path and appending the edge vx in the beginning. Eventually, if $u_1 = v$ and $u_2 = x$ we repeat the previous steps, replacing x with y. $\square$

Lemma 6. *Let G be a connected AP graph of even order that satisfies inequality (1). If there exist two nonadjacent vertices x, y of G such that $d(x) + d(y) \leq 5$, then G is traceable.*

Proof. Like in the proof of Lemma 3, we argue by contradiction considering $G' = G - \{x, y\}$. A similar edge counting gives us that there are only $|G'| - 3$ edges missing from K_{n-2}. By Lemma 3 at least one of x, y must have 3 neighbours in G', say x, so we have three pairs of neighbours of x. Lemma 5 guarantees that we can choose a pair for which there is a Hamiltonian path in G' which we can extend to a Hamiltonian cycle in $G - y$ and then to a Hamiltonian path in G. $\square$

The next lemma was proved in [5] and follows from a result of Woodall [9].

Lemma 7. ([5]) *For any positive integer δ, if $n = |G|$ and*

$$||G|| > \binom{n-\delta}{2} + \binom{\delta+1}{2}$$

then $c(G) > n - \delta$, where $c(G)$ is the length of the longest cycle in G.

We also introduce a *sun with two rays* graph, which consists of a cycle with appended two hanging vertices u_1, u_2. We demand that each vertex on a cycle has degree 2 or 3. If x_1, x_2 are the neighbours of, respectively, u_1 and u_2 on the cycle and between x_1 and x_2 there are a and b vertices on the cycle (on both sides, therefore the whole graph has $a + b + 4$ vertices), then we denote this graph as $\mathrm{Sun}(a, b)$. Kalinowski, Pilśniak, Woźniak and Zioło [6] characterized all on-line AP suns with two rays what is summarized in Table 1.

a	b
0	arbitrary
1	$\equiv 0 \pmod 2$
2	$\not\equiv 3 \pmod 6$, 3,9,21
3	$\equiv 0 \pmod 2$
4	$\equiv$ 2,4 (mod 6), $[4, 19] \setminus \{15\}$
5	$\equiv$ 2,4 (mod 6), 6, 18
6	6,7,8,10,11,12,14,16
7	8,10,12,14,16
8	8,9,10,11,12
9	10,12

Table 1: On-line AP suns

At the end of this section we state a result of Li and Ning [8]. By B_n^k we denote a graph obtained from $K_{n,n}$ by deleting all edges of its subgraph $K_{n-k,k}$.

Lemma 8. ([8]) *Let G be a balanced bipartite graph of order $2n$. If $\delta(G) \geq k \geq 1, n \geq 2k + 1$ and*

$$||G|| > n(n - k - 1) + (k + 1)^2$$

then G is Hamiltonian unless G is a subgraph of B_n^k.

Corollary 9. *Let G be a balanced bipartite graph of order $2n$. If $\delta(G) \geq 1, n \geq 3$ and*

$$||G|| \geq n(n - 2) + 4$$

then G is traceable unless G is a subgraph of B_n^1.

Proof. We add any edge e to satisfy a strict inequality in Lemma 8 for $k = 1$. If $G + e$ is Hamiltonian, then G is traceable, otherwise $G + e$ is a subgraph of B_n^1. Such a subgraph is just any balanced bipartite graph with a vertex of degree 1, but we have $\delta(G) \geq 1$, so in that case G is also a subgraph of B_n^1. □

3. Proof of Theorem 1

By definition, if a graph is on-line AP, then it is also AP. Let G be connected AP graph of order n, satisfying inequality (1). We also assume that G is not traceable, since otherwise G would be on-line AP.

By Lemma 7 with $\delta = 3$, there is a cycle C of length at least $n-2$ in G. If C has length $n-1$ or n, then G is traceable. Therefore we assume that C has length exactly $n-2$. Denote the vertices outside C as u, v, and assume $\deg(u) \leq \deg(v)$. Then $uv \notin E(G)$, otherwise G would be traceable.

In this proof we first consider the odd values of n and resolve them using Table 1. Then for any of the even orders of G we find a set of edges between the vertices of C with the following property: for any edge e in this set, if $e \in E$, then G is traceable. We prove that if this set induces a clique of order $\frac{n}{2}$, then the case is completed. Moreover, note that if this set is too large, then we can have a contradiction with the assumptions.

If n is odd, then G has a Sun(a,b) as a spanning subgraph with $a+b$ being odd and equal to at most 9. A short look at Table 1 shows that any such sun is on-line AP, therefore G has an on-line AP spanning subgraph and so it is on-line AP itself. In the rest of the proof we assume that n is even.

Let $C = v_1v_2\ldots v_{n-2}v_1$. In the further reasoning we say that we *exclude* an edge e if the existence of e in G would imply either traceability of G or a contradiction with the assumptions. We claim the following.

Claim 1. *Let v_i, v_j have both a neighbour outside the cycle C, not necessarily the same one. If $v_{i+1}v_{j+1} \in E(G)$ or $v_{i-1}v_{j-1} \in E(G)$, then G is traceable.*

Proof. If $vv_i, uv_j \in E$, then the Hamiltonian path would be: $uv_i \ldots v_{j+1}v_{i+1} \ldots v_jv$. For $v_{i-1}v_{j-1}$ reverse the ordering of the cycle. If $vv_i, vv_j \in E$ or $uv_i, uv_j \in E$, then the above path becomes a cycle of length $n-1$ that can be extended to a Hamiltonian path. □

By Lemma 6, we deduce that $\deg(u) + \deg(v) \geq 6$ and $\deg(v) \geq 3$. Another look at Table 1 reveals that if $|C| \in \{6, 8, 10, 12\}$ (which means that Sun(a, b), with $a + b \in \{4, 6, 8, 10\}$, is as a spanning subgraph of G) and if $vv_1, uv_{2k} \in E$ for some k (which means that both of a, b are even), then the whole graph is on-line AP since it contains an on-line AP sun as a spanning subgraph. Hence the only possible edges from u, v to the cycle C have ends with indices of the same parity.

For $n = 8$, we have $|C| = 6$ and to avoid the neighbours of v being consecutive, we can only choose them to be (up to isomorphism) v_1, v_3 and v_5. But then by Claim 1, we exclude the edges v_2v_4, v_2v_6 and v_4v_6 which gives us an independent set $\{u, v, v_2, v_4, v_6\}$ of size 5 and makes it impossible for G to be AP since there cannot be a perfect matching.

For $n = 10$ the cycle C has length 8, so v has to have 3 neighbours whose indices are consecutive numbers of the same parity. Without loss of generality let the neighbours of v be v_1, v_3 and v_5. Then by Claim 1, the edges $v_2v_4, v_2v_6, v_4v_6, v_8v_2, v_8v_4$ cannot appear in the graph. Now, if $v_6v_8 \notin E$, then we get an independent set of order 6 which again gives a contradiction. Otherwise, note that we can assume that $\deg(u) = \deg(v) = 3$ and that all the neighbours of u, v are common, since connecting u or v with v_7 immediately excludes the edge v_6v_8. Then we exclude the edges v_2v_7 and

v_4v_7, otherwise there would exist Hamiltonian paths $uv_1vv_5v_4v_3v_2v_7v_6v_8$ and $uv_5vv_1v_2v_3v_4v_7v_6v_8$, respectively. In total, we have excluded 7 edges which gives

$$||G|| \leq d(u) + d(v) + ||K_8|| - 7 = 6 + \binom{8}{2} - 7 = 27,$$

which is less than in our assumption, i.e., $\binom{7}{2} + 7 = 28$.

Claim 2. *If G has an independent set of size $\frac{n}{2}$, for $n \in \{12, 14\}$, then either G is traceable or G is not AP.*

Proof. Let S be such an independent set. If there is a superset of S that is also independent, then G has no perfect matching and is not AP. Therefore assume that any vertex in $V \setminus S$ has a neighbour in S and since G is connected, every vertex in S has a neighbour in $V \setminus S$.

For a moment we ignore the edges inside $V \setminus S$. The number of edges between S and $V \setminus S$ is at least

$$||G|| - ||K_{\frac{n}{2}}|| \geq \binom{n-3}{2} + 7 - \binom{\frac{n}{2}}{2} = \frac{3}{8}n^2 - \frac{13}{4}n + 13.$$

Since for $n \geq 12$ the inequality $\frac{3}{8}n^2 - \frac{13}{4}n + 13 \geq \frac{n}{2}(\frac{n}{2} - 2) + 4$ is satisfied, then, by Corollary 9, to complete the proof of the claim we need to consider the case when there is a vertex $v \in V$ with only one neighbour in the other set of bipartition. For $n = 12$ we have at most 28 edges between S and $V \setminus S$, and when we exclude v, there are at most 27 edges between them, out of the maximum of 30 edges in a complete bipartite graph $K_{5,6}$. Then we can easily find a Hamiltonian path starting from v. The same situation is for $n = 14$, when excluding v yields 40 out of maximum 42 edges in $K_{6,7}$. □

Now we can consider $n = 12$. The cycle C has length 10. Recall that $\deg(u) + \deg(v) \geq 6$, $\deg(v) \geq 3$, and if $vv_1 \in E$, then u and v can have edges only to vertices of odd indices. There are 5 such vertices so if there are 4 neighbours of u, v on the cycle, then their indices are four consecutive odd numbers (modulo 10), without loss of generality v_1, v_3, v_5, v_7. Thus by Claim 1 there is an independent set $\{v_2, v_4, v_6, v_8, u, v\}$ of size 6 which using Claim 2 finishes the case. So there are exactly three neighbours of u, v on C and they are common neighbours. This gives two cases up to symmetry: when these neighbours are either v_1, v_3, v_5 or v_1, v_3, v_7.

It suffices to show that there are eight edges that cannot be in E since otherwise G would be traceable, because

$$d(u) + d(v) + ||K_{10}|| - 8 = 42 < 43 = \binom{9}{2} + 7.$$

1. The common neighbours of u, v are v_1, v_3, v_5. Then by Claim 1, we exclude edges v_2v_4, v_2v_6, v_4v_6, v_2v_{10}, v_4v_{10}. If $v_6v_{10} \notin E(G)$, we get an independent set of size $6 = \frac{12}{2}$. Otherwise, we exclude the following edges.

edge	Hamiltonian path
v_2v_9	$uv_1vv_5v_4v_3v_2v_9v_8v_7v_6v_{10}$
v_4v_9	$uv_5vv_1v_2v_3v_4v_9v_8v_7v_6v_{10}$
v_2v_7	$uv_1vv_5v_4v_3v_2v_7v_8v_9v_{10}v_6$

Table 2

2. The common neighbours of u, v are v_1, v_3, v_7. Then by Claim 1, we exclude edges v_2v_4, v_2v_6, v_2v_8, v_2v_{10}, v_4v_8, v_6v_{10}. Next we exclude v_6v_8 using a path $uv_7vv_1v_2v_3v_4v_5v_6v_8v_9v_{10}$, which is a seventh excluded edge. The last one is either v_8v_{10} or v_6v_9 since both of them would enable a path $uv_7vv_1v_2v_3v_4v_5v_6v_9v_8v_{10}$.

The remaining case is $n = 14$. The cycle C has length 12. There are six vertices with odd indices, so if five of them are the neighbours of u or v, then they represent five consecutive odd numbers modulo 12, without loss of generality, v_1, v_3, v_5, v_7 and v_9. Then by Claim 1, the set $\{v_2, v_4, v_6, v_8, v_{10}, u, v\}$ is independent and by Claim 2 this case can be omitted. Then each of u, v has degree at most 4 and at least 2. Moreover, they must have at least two common neighbours.

Like for smaller n, if we exclude 13 edges, then we are done since

$$\deg(u) + \deg(v) + ||K_{12}|| - 13 \leq 8 + 66 - 13 = 61 < 62 = \binom{11}{2} + 7.$$

Up to symmetry, we have the following cases, setting first two common neighbours and then an additional neighbour for v. The vertex u can be adjacent to two consecutive odd-indexed vertices, say v_1, v_3 (1), then the neighbours of v can be either v_1, v_3, v_5 (1a) or v_1, v_3, v_7 (1b). If u is adjacent to v_1, v_5 (2), then the possibilities for the neighbours of v are v_1, v_3, v_5 (2a) or v_1, v_5, v_7 (2b) or v_1, v_5, v_9 (2c). Finally, if the neighbours of u are v_1, v_7 (3), then the only option for v modulo rotations and symmetries is v_1, v_3, v_7 (3a). To make the proof shorter, we invoke cases (1a), (2b) and (2c) in case (2a) as well as (1a) and (3a) in case (1b). We have enumerated this now to show that none of the cases refers to itself.

(1a) The neighbours of u are v_1, v_3, the neighbours of v are v_1, v_3, v_5 (see Figure 1). By Claim 1, we exclude edges v_2v_4, v_2v_6, v_4v_6, v_2v_{12}, v_4v_{12}. Then we exclude one of v_4v_7 or v_6v_8, since if they were both in E, we would have a path $v_{12}v_{11}v_{10}v_9v_8v_6v_7v_4v_5vv_3uv_1v_2$. Let us suppose that $v_4v_{10} \in E$. Then we can exclude seven edges (in Table 3 below), that gives a total of 13 which is too many. Table 3 should be read as follows: a row with an edge e and a path P means that if $v_4v_{10} \in E$ and $e \in E$, then G is traceable and has a Hamiltonian path P. A row with two edges e, f means that if $v_4v_{10} \in E$ and both $e, f \in E$, then P is a Hamiltonian path in G (so either e or f is excluded).

We repeat these arguments for edges v_2v_{10}, v_6v_{12}, v_6v_{10} and $v_{10}v_{12}$, each time decreasing the number of edges to exclude. This yields an independent set $\{v_2, v_4, v_6, v_{10}, v_{12}, u, v\}$ of size 7, which by Claim 2 completes the case.

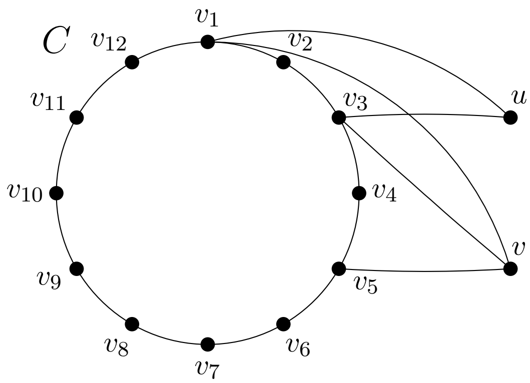

Figure 1: Graph G in case (1a)

edge	Hamiltonian path
v_6v_{11}	$v_{12}v_{11}v_6v_7v_8v_9v_{10}v_4v_5vv_3uv_1v_2$
v_6v_{12}	$v_{11}v_{12}v_6v_7v_8v_9v_{10}v_4v_5vv_3uv_1v_2$
v_9v_{12}	$v_6v_7v_8v_9v_{12}v_{11}v_{10}v_4v_5vv_3uv_1v_2$
v_2v_9	$v_6v_7v_8v_9v_2v_1uv_3vv_5v_4v_{10}v_{11}v_{12}$
v_2v_{11}	$v_6v_7v_8v_9v_{10}v_4v_5vv_3uv_1v_2v_{11}v_{12}$
v_4v_{11}	$v_{12}v_{11}v_4v_{10}v_9v_8v_7v_6v_5vv_3uv_1v_2$
v_4v_9	$v_{12}v_{11}v_{10}v_4v_9v_8v_7v_6v_5vv_3uv_1v_2$

Table 3: Excluding v_4v_{10}

edge	Hamiltonian path
v_9v_{12}	$v_6v_7v_8v_9v_{12}v_{11}v_{10}v_2v_1uv_3vv_5v_4$
v_6v_{12}	$v_9v_8v_7v_6v_{12}v_{11}v_{10}v_2v_1uv_3vv_5v_4$
v_6v_{11}	$v_{12}v_{11}v_6v_7v_8v_9v_{10}v_2v_1uv_3vv_5v_4$
v_2v_9	$v_{12}v_{11}v_{10}v_2v_9v_8v_7v_6v_5v_4v_3vv_1u$
v_2v_{11}	$v_{12}v_{11}v_2v_{10}v_9v_8v_7v_6v_5v_4v_3vv_1u$
v_4v_{11}	$v_6v_7v_8v_9v_{10}v_2v_1uv_3vv_5v_4v_{11}v_{12}$

Table 4: Excluding v_2v_{10}

edge	Hamiltonian path
v_4v_{11}	$v_{10}v_9v_8v_7v_6v_{12}v_{11}v_4v_5vv_3uv_1v_2$
v_4v_9	$v_8v_7v_6v_{12}v_{11}v_{10}v_9v_4v_5vv_3uv_1v_2$
v_2v_7	$v_8v_9v_{10}v_{11}v_{12}v_6v_7v_2v_1uv_3vv_5v_4$
v_2v_9	$v_{10}v_{11}v_{12}v_6v_7v_8v_9v_2v_1uv_3vv_5v_4$
v_2v_{11}	$v_{10}v_9v_8v_7v_6v_{12}v_{11}v_2v_1uv_3vv_5v_4$

Table 5: Excluding v_6v_{12}

edge	Hamiltonian path
v_2v_9	$v_{12}v_{11}v_{10}v_6v_7v_8v_9v_2v_1uv_3vv_5v_4$
v_4v_9	$v_{12}v_{11}v_{10}v_6v_7v_8v_9v_4v_5vv_3uv_1v_2$
v_6v_{11} or v_2v_7	$v_{12}v_{11}v_6v_{10}v_9v_8v_7v_2v_1uv_3vv_5v_4$
v_4v_{11} or v_9v_{12}	$v_{10}v_6v_7v_8v_9v_{12}v_{11}v_4v_5vv_3uv_1v_2$

Table 6: Excluding v_6v_{10}

edge	Hamiltonian path
v_2v_{11}	$v_6v_7v_8v_9v_{10}v_{12}v_{11}v_2v_1uv_3vv_5v_4$
v_4v_{11}	$v_6v_7v_8v_9v_{10}v_{12}v_{11}v_4v_5vv_3uv_1v_2$
v_6v_{11} or v_2v_9	$v_8v_7v_6v_{11}v_{12}v_{10}v_9v_2v_1uv_3vv_5v_4$

Table 7: Excluding $v_{10}v_{12}$

(1b) The neighbours of u are v_1, v_3, the neighbours of v are v_1, v_3, v_7. Claim 1 allows us to exclude edges v_2v_4, v_2v_6, v_2v_8, v_2v_{12}, v_4v_8, v_6v_{12}. Then v_4v_{12} can be excluded using a path $v_2v_1uv_3vv_7v_6v_5v_4v_{12}v_{11}v_{10}v_9v_8$. Consider now an additional sixth edge from $\{u, v\}$ to the cycle C. If it ends in v_5 or v_{11}, then we can use the previous case (1a). If the end is v_7, the case (3a) can be used. So we can assume that there is an edge from u or v to v_9, which by Claim 1 excludes the edges v_8v_{12}, v_6v_8, v_8v_{10}, v_2v_{10}, v_4v_{10}. We finish the case, removing the 13th edge, which is one of v_2v_5, v_4v_6, with a Hamiltonian path $v_2v_5v_4v_6v_7v_8v_9v_{10}v_{11}v_{12}v_1uv_3v$.

(2a) The neighbours of u are v_1, v_5, the neighbours of v are v_1, v_3, v_5. Due to Claim 1, we exclude edges v_2v_4, v_2v_6, v_4v_6, v_2v_{12}, v_4v_{12}. Like in the previous case, we consider the additional edge from u, v. If its end is in v_3, then we use case (1a), for v_7 and v_{11} we use (2b), and for v_9 we use (2c).

(2b) The neighbours of u are v_1, v_5, the neighbours of v are v_1, v_5, v_7. It follows from Claim 1 that the edges v_2v_6, v_2v_8, v_4v_6, v_4v_{12}, v_6v_8, v_6v_{12} can be excluded. Then we exclude the edge v_4v_8 with a Hamiltonian path $v_{12}v_{11}v_{10}v_9v_8v_4v_3v_2v_1uv_5v_6v_7v$ and the edge v_2v_{12} with a path $v_4v_3v_2v_{12}v_{11}v_{10}v_9v_8v_7vv_1uv_5v_6$. Now if we exclude the edges v_8v_{12} and v_2v_4, then we end up with an independent set $\{v_2, v_4, v_6, v_8, v_{12}, u, v\}$ of size 7 on vertices. We present in the tables below how to accomplish it, like in the case (1a).

edge	Hamiltonian path
v_6v_{11}	$v_{10}v_9v_8v_{12}v_{11}v_6v_7vv_5v_4v_3v_2v_1u$
v_6v_9	$v_{10}v_{11}v_{12}v_8v_9v_6v_7vv_5v_4v_3v_2v_1u$
v_4v_9	$v_{10}v_{11}v_{12}v_8v_9v_4v_3v_2v_1uv_5v_6v_7v$
v_4v_{11}	$v_{10}v_9v_8v_{12}v_{11}v_4v_3v_2v_1uv_5v_6v_7v$
v_2v_{11}	$v_6v_7vv_5v_4v_3v_2v_{11}v_{10}v_9v_8v_{12}v_1u$

Table 8: Excluding v_8v_{12}

edge	Hamiltonian path
v_3v_6	$v_4v_2v_3v_6v_7vv_5uv_1v_{12}v_{11}v_{10}v_9v_8$
v_3v_8	$v_{12}v_{11}v_{10}v_9v_8v_3v_4v_2v_1uv_5v_6v_7v$
v_3v_{12}	$v_4v_2v_3v_{12}v_{11}v_{10}v_9v_8v_7vv_1uv_5v_6$
v_8v_{10} or v_6v_9	$v_{12}v_{11}v_{10}v_8v_9v_6v_7vv_5v_4v_3v_2v_1u$

Table 9: Excluding v_2v_4

(2c) The neighbours of u are v_1, v_5, the neighbours of v are v_1, v_5, v_9. By Claim 1, we exclude the edges v_2v_6, v_2v_{10}, v_4v_8, v_4v_{12}, v_6v_{10} and v_8v_{12}. The remaining seven edges to exclude are listed in Table 10 below.

edge	Hamiltonian path
v_3v_6	$v_4v_2v_3v_6v_7vv_5uv_1v_{12}v_{11}v_{10}v_9v_8$
v_3v_8	$v_{12}v_{11}v_{10}v_9v_8v_3v_4v_2v_1uv_5v_6v_7v$
v_3v_{12}	$v_4v_2v_3v_{12}v_{11}v_{10}v_9v_8v_7vv_1uv_5v_6$
v_3v_6	$v_4v_2v_3v_6v_7vv_5uv_1v_{12}v_{11}v_{10}v_9v_8$
v_3v_8	$v_{12}v_{11}v_{10}v_9v_8v_3v_4v_2v_1uv_5v_6v_7v$
v_3v_{12}	$v_4v_2v_3v_{12}v_{11}v_{10}v_9v_8v_7vv_1uv_5v_6$
v_8v_{10} or v_6v_9	$v_{12}v_{11}v_{10}v_8v_9v_6v_7vv_5v_4v_3v_2v_1u$

Table 10

(3a) The neighbours of u are v_1, v_7, the neighbours of v are v_1, v_3, v_7. We use Claim 1 to exclude the edges v_2v_4, v_2v_6, v_2v_8, v_2v_{12}, v_4v_8 and v_6v_{12}. Next we exclude the edge v_4v_{12} using the path $uv_1v_2v_3vv_7v_6v_5v_4v_{12}v_{11}v_{10}v_9v_8$ and the edge v_6v_8 using the path $uv_7vv_1v_2v_3v_4v_5v_6v_8v_9v_{10}v_{11}v_{12}$. Now we miss only the edges v_4v_6 and v_8v_{12} to have an independent set $\{v_2, v_4, v_6, v_8, v_{12}, u, v\}$ of size 7. We exclude them successively in the tables below.

edge	Hamiltonian path
v_2v_9	$uv_1vv_7v_6v_5v_4v_3v_2v_9v_8v_{12}v_{11}v_{10}$
v_4v_9	$v_2v_3vv_1uv_7v_6v_5v_4v_9v_8v_{12}v_{11}v_{10}$
v_6v_9	$v_2v_3vv_1uv_7v_8v_{12}v_{11}v_{10}v_9v_6v_5v_4$
v_2v_{11}	$uv_1vv_7v_6v_5v_4v_3v_2v_{11}v_{12}v_8v_9v_{10}$
v_4v_{11}	$uv_7vv_3v_2v_1v_{12}v_8v_9v_{10}v_{11}v_4v_5v_6$

Table 11: Excluding v_8v_{12}

edge	Hamiltonian path
v_2v_5	$vv_3v_2v_5v_4v_6v_7v_8v_9v_{10}v_{11}v_{12}v_1u$
v_5v_8	$uv_7v_6v_4v_5v_8v_9v_{10}v_{11}v_{12}v_1v_2v_3v$
v_5v_{12}	$uv_1v_2v_3v_4v_6v_5v_{12}v_{11}v_{10}v_9v_8v_7v$
v_2v_{10} or v_9v_{12}	$uv_1vv_3v_2v_{10}v_{11}v_{12}v_9v_8v_7v_6v_5v_4$

Table 12: Excluding v_4v_6

Since all the cases has been completed, the proof is now finished.

□

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