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Dense on-line arbitrarily partitionable graphs *

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Abstract

A graph G of order n is called *arbitrarily partitionable* (AP, for short) if, for every sequence $(n_1, \ldots, n_k)$ of positive integers with $n_1 + \ldots + n_k = n$, there exists a partition $(V_1, \ldots, V_k)$ of the vertex set $V(G)$ such that V_i induces a connected subgraph of order n_i, for $i = 1, \ldots, k$. In this paper we consider the on-line version of this notion, defined in a natural way.

We prove that if G is a connected graph such that $\alpha(G) \leq \lceil \frac{n}{2} \rceil$ and the degree sum of any pair of non-adjacent vertices is at least $n-3$, then G is on-line arbitrarily partitionable except for two graphs of small orders. We also prove that if G is a connected graph of order n and size $\|G\| > \binom{n-3}{2} + 6$, then G is on-line AP unless n is even and G is a spanning subgraph of a unique exceptional graph. These two results imply that AP dense graphs satisfying one of the above two assumptions are also on-line AP. This is in contrast to sparse graphs where only few AP graphs are also on-line AP. While proving our main results, we also obtain some sufficient conditions for a graph to be traceable.

Keywords: partitions of graphs, traceable graph, Erdős-Gallai condition, Ore condition, perfect matching.

Mathematics Subject Classifications: 05C70, 05C45, 05C40.

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1 Introduction

We use standard graph theory terminology and notation. The number of edges of a graph G is called the *size* of G and is denoted by $\|G\|$. A graph G is called *traceable* if it contains a *Hamiltonian path*, i.e. a path through all vertices of G. By $c(G)$ we denote the *circumference* of a graph G, i.e. the length of a longest cycle. If C is a cycle with a given orientation and $x \in V(C)$, then by x^+ and x^- we denote a successor and a predecessor of x along the orientation of C. We also use the notation

$$\sigma_2(G) = \min\{d(x) + d(y) \,:\, xy \notin E(G)\}.$$

If G_1 and G_2 are two graphs with disjoint vertex sets, then by $G_1 \vee G_2$ we denote their *join*, that is a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \,:\, u \in V(G_1), v \in V(G_2)\}$.

Let us now introduce some terminology for the problem we deal with. If $G = (V, E)$ is a graph of order n, then a sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is called *admissible for* G if $n_1 + \ldots + n_k = n$. Such an admissible sequence τ is said to be *realizable in* G if the vertex set V can be partitioned into k parts $(V_1, \ldots, V_k)$ such that $|V_i| = n_i$ and the subgraph $G[V_i]$ induced by V_i is connected, for every $i = 1, \ldots, k$. We say that G is *arbitrarily partitionable* (AP, for short) if every admissible sequence τ for G is realizable in G.

The notion of AP graphs was introduced by Barth, Baudon and Puech in [1] (and independently by Horňák and Woźniak in [12]), motivated by a problem concerning graphs modelling parallel systems (parallel computing, networks of workstations, etc.), considered as networks connecting different computing resources. Suppose there are k users, where the i-th one needs n_i resources from our network. The subgraph induced by the set of resources attributed to each user should be connected and each resource should be attributed to one user. So we are seeking a realization of the sequence $\tau = (n_1, \ldots, n_k)$ in this graph. Suppose that we want to do it for any number of users and any sequence of request. Thus, such a network should be an AP graph.

The general concept of arbitrarily partitionable graphs, sometimes also called *arbitrarily vertex decomposable* [12] or *fully decomposable* [7] or just *decomposable* [1], has spawned some thirty papers and is still developing (examples of recent papers are [3, 4, 7, 15, 20]). Here we quote only the results we directly use in this paper.

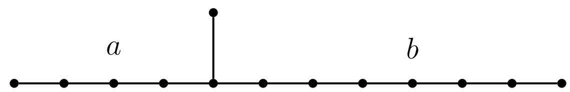

Figure 1: Cat(a, b) with $a = 5$ and $b = 8$

By Cat(a, b), where $2 \leq a \leq b$, we denote a caterpillar with three leaves obtained from the star $K_{1,3}$ by substituting two of its edges by paths of orders a and b, respectively. Figure 1 shows Cat$(5, 8)$. One of the earliest results about AP graphs is the following one proved independently by the authors of this concept.

Theorem 1 [1, 12] *The caterpillar* Cat(a, b) *is AP if and only if a and b are relatively prime.*

A *sun with r rays* is a graph of order $n \geq 2r$ with r independent hanging edges, called *rays*, whose deletion yields a cycle C_{n-r}. A sun with two rays (cf. Figure 2) such that the deletion of vertices of degree three divides C_{n-r} into two paths of orders a and b, where $0 \leq a \leq b$, is denoted by Sun(a, b).

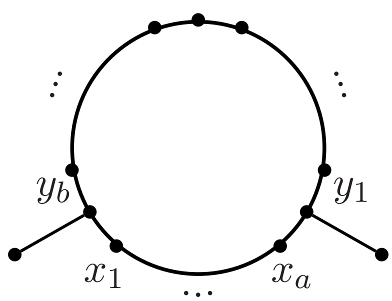

Figure 2: Sun(a, b) with two rays

In [13], we found all AP suns with at most three rays. Our result for suns with two rays follows.

Theorem 2 [13] Sun(a, b) *is AP if and only if at least one of the numbers a, b is even.*

a	b
2	$\equiv 1 \pmod 2$
3	$\equiv 1, 2 \pmod 3$
4	$\equiv 1 \pmod 2$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod 6$
7	8, 9, 11, 13, 15
8	11, 19
9	11
10	11
11	12

Table 1: Values a, b such that $\mathrm{Cat}(a, b)$ is on-line AP

The definition of AP graphs has many variations. One of them, even more natural from the point of view of applications to computer science, is the following concept of on-line arbitrarily partitionable graphs introduced by Horňák, Tuza and Woźniak in [11]. The definition is natural. We are not given a whole admissible sequence at the beginning but we get its elements one by one, and each time we have to choose a connected subgraph of prescribed order, having no possibility to change this choice later. If this procedure can be accomplished for any admissible sequence, then the graph is called *on-line arbitrarily partitionable* (on-line AP, for short).

Horňák, Tuza and Woźniak [11] characterized all on-line AP trees.

Theorem 3 [11] *A tree T is on-line AP if and only if T is either a path or a caterpillar* $\mathrm{Cat}\,(a, b)$ *with a and b given in Table* 1 *or the tripode* $S(3, 5, 7)$*.*

It follows that every on-line AP tree has at most three vertices. This is in contrast with the following result of Barth, Fournier and Ravaux [2].

Theorem 4 [2] *For every positive integer l there exists an AP tree with l leaves.*

Moreover, even for trees with three leaves, on-line AP trees constitute a small subclass of AP ones. For instance, $\mathrm{Cat}(5, b)$ is AP for every $b \not\equiv 0 \pmod 5$ by Theorem 1, but it is never on-line AP for $b > 20$. That means that on-line AP trees constitute a relatively small subset of the set of all AP

a	b
0	arbitrary
1	$\equiv 0 \pmod 2$
2	$\not\equiv 3 \pmod 6$, 3, 9, 21
3	$\equiv 0 \pmod 2$
4	$\equiv 2, 4 \pmod 6$, $[4, 19] \setminus \{15\}$
5	$\equiv 2, 4 \pmod 6$, 6, 18
6	6, 7, 8, 10, 11, 12, 14, 16
7	8, 10, 12, 14, 16
8	8, 9, 10, 11, 12
9	10, 12

Table 2: Values a, b such that $\mathrm{Sun}(a, b)$ is on-line AP

trees. A similar situations holds for suns. In [14], we characterized all on-line AP suns with any number of rays. We showed that the number of rays in an on-line AP sun does not exceed four, and we proved the following for suns with two rays.

Theorem 5 [14] $\mathrm{Sun}\,(a, b)$ *is on-line AP if and only if* a *and* b *take values given in Table* 2*.*

Observe that, in particular, there are only finitely many on-line AP suns $\mathrm{Sun}(a, b)$ with both a and b greater than 5. It follows that among sparse graphs, like trees and suns, AP graphs rather rarely are also on-line AP. This is not the case for dense graphs as we show in this paper.

The following obvious fact plays a fundamental role in our study.

Proposition 6 [11] *Every traceable graph is on-line AP.*

Some sufficient conditions for a graph to be traceable have been weakened to obtain sufficient conditions for a graph to be AP. The first one is the well-known Ore condition.

Theorem 7 *If* $\sigma_2(G) \geq n - 1$*, then* G *is traceable.*

This condition was first adapted for AP graphs by Marczyk, among others in [18].

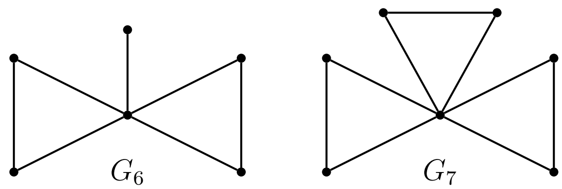

Figure 3: Two exceptional non-AP graphs with $\sigma_2 \geq n-3$ and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$

Theorem 8 [18] *If G is a connected graph with $\sigma_2(G) \geq n-3$ and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, then G is AP or G is one of two graphs G_6, G_7 depicted in Figure 3.*

Note that the connectivity of G and the inequality $\alpha(G) \leq \lceil \frac{n}{2} \rceil$ are obvious necessary conditions for a graph G to be AP.

Another sufficient condition for traceability follows from a paper [9] of Erdős and Gallai (cf. e.g. [15]).

Theorem 9 *If G is a connected graph such that*

$$\|G\| > \binom{n-2}{2} + 2,$$

then G is traceable.

Recently, Kalinowski, Pilśniak, Schiermeyer and Woźniak in [15] proved the following.

Theorem 10 [15] *If G is a connected graph of order $n \geq 22$ and size*

$$\|G\| > \binom{n-4}{2} + 12,$$

then G is AP unless G is a spanning subgraph of one of the graphs depicted in Figure 4 .

The aim of this paper is to prove the following result which is a consequence of Theorem 18, the main result of Section 2, and Theorem 24, the main result of Section 3.

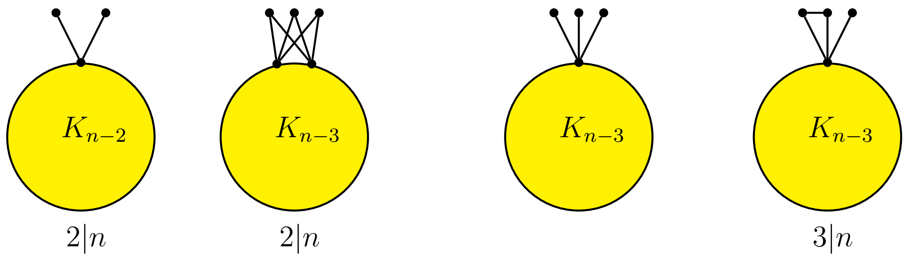

Figure 4: Four graphs such that every non-AP graph G of order $n \geq 22$ and size $\|G\| > \binom{n-4}{2} + 12$ is a spanning subgraph of one of them (below each graph, requirements on the order n are given)

Theorem 11 *If G is a graph of order n such that*

$$\sigma_2(G) \geq n - 3$$

or

$$\|G\| > \binom{n-3}{2} + 6,$$

then G is AP if and only if G is on-line AP.

2 Ore-type condition

In the proof of Theorem 14, which is the main result of this section, we make use of the following two classical results.

Theorem 12 (Posa [19]) *Let G be a connected graph of order $n \geq 3$ such that*

$$\sigma_2(G) \geq s.$$

Then G contains a path of length s or G is Hamiltonian.

Theorem 13 (Bermond [6], Linial [17]) *Let G be a 2-connected graph such that*

$$\sigma_2(G) \geq s.$$

Then G contains a cycle of length at least s or G is Hamiltonian.

Now, from the exactly the same assumptions as in Theorem 8 we derive a stronger conclusion.

Theorem 14 *If G is a connected graph with $\sigma_2(G) \geq n-3$ and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, then G is on-line AP or $G \in \{G_1, G_2\}$.*

Proof. Assume first that $G = (V, E)$ is 2-connected. It follows from Theorem 13 that the circumference $c(G)$ is at least $n-3$. If $c(G) \geq n-1$, then G is traceable, and hence on-line AP.

Suppose first that $c(G) = n-2$. Let C be a cycle of length $n-2$ in G, and let u, v be the two vertices of G outside C. If $uv \in E$, then G is traceable. Then suppose $uv \notin E$. Hence, $d(u) + d(v) \geq n-3$. Assume $d(v) \leq d(u)$. As C is a longest cycle, no two consecutive vertices of C can be neighbours of the same vertex outside C. Therefore, both vertices u and v are of degree at most $\frac{n-2}{2}$. Thus, $\frac{n-4}{2} \leq d(v) \leq d(u) \leq \frac{n-2}{2}$. Suppose, contrary to the claim, that G is not on-line AP, whence also not traceable. Therefore, $x^-, x^+ \notin N(u) \cup N(v)$ whenever $x \in N(u) \cup N(v)$.

If n is odd, then $d(v) = d(u) = \frac{n-3}{2}$. It follows that $N(u) = N(v)$ for $n \geq 7$. Then there exists a vertex $x \in V(C)$ such that $x^-u, x^+v \in E$. So G is spanned by $\mathrm{Sun}(1, n-5)$ which is on-line AP for odd n by Theorem 5, whence G is also on-line AP. If $n = 5$, then G has to be traceable for it is a bull (a graph isomorphic to a letter A) what can be easily seen.

Let n be even. Hence $d(u) = \frac{n-2}{2}$, i.e. u is adjacent to every second vertex of C. Then clearly, $N(v) \subseteq N(u)$. The set $A = \{x^+ : xu \in E\} \cup \{u, v\}$ is independent, and $|A| = \frac{n}{2} + 1$, contrary to the assumption.

Now, consider the case $c(G) = n-3$. Let C be a longest cycle in G and let $V \setminus V(C) = \{u, v, w\}$. If $G[\{u, v, w\}]$ is connected, then G is traceable. Suppose then that $E(G[\{u, v, w\}]) = \{uv\}$. Hence, $d(w) \leq \frac{n-3}{2}$ for C is a longest cycle, so w cannot be adjacent to two successive vertices on C. Consequently, $d(u), d(v) \geq \frac{n-5}{2} + 1$ since $\sigma_2(G) \geq n-3$. Therefore, it is easily seen that there exists a vertex $x \in V(C)$ such that either $x^- \in N(u)$ and $x^+ \in N(v)$ or $x^+ \in N(u)$ and $x^- \in N(v)$. But then G would contain a cycle longer than C.

Suppose now that the vertices u, v, w comprise an independent set. Then at least two of them, say u and v, are of degree at least $\frac{n-3}{2}$ since $\sigma_2(G) \geq n-3$. But with our assumptions this is possible only when $d(w) = d(u) = d(v) = \frac{n-3}{2}$ and $N(u) = N(v) = N(w)$. Then G has an independent set $A = \{x^+ :\ x \in N(u)\} \cup \{u, v, v\}$ of cardinality $\frac{n-3}{2} + 3$ which is greater than $\lceil \frac{n}{2} \rceil$, a contradiction.

Now, assume that G has a cut vertex z, and let s be the number of connected components of $G-z$. Suppose first that $s \geq 3$. Let P be a longest path in G. By Theorem 12, the order of P is at least $n-2$, i.e. there are at most two vertices outside P. Then it is easy to see that $z \in V(P)$ and $s = 3$ since $\sigma_2 \geq n-3$. If P is of order $n-2$, then the two vertices u, v outside P are of degree 2, and $G - z$ has three connected components with vertex sets V_1, V_2 and $V_3 = \{u, v\}$. For every $x \in V_1 \cup V_2$ we have $d(x) + d(u) \geq n - 3$, hence $d(x) \geq n-5$. On the other hand, if $x \in V_i$, then $d(x) \leq |V_i|$, $i = 1, 2$, since P is a longest path in G. Therefore, $|V_1| = |V_2| = 2$ and $d(x) = 2$ for each $x \in V_1 \cup V_2$. That is, G is isomorphic to G_7 of Figure 3.

Then suppose there is only one vertex outside P. Hence, $s = 3$. Analogously as in the previous case, we infer that $|V_1| = |V_2| = 2$ and $d(x) = 2$ for each $x \in V_1 \cup V_2$. Thus, G is isomorphic to G_6.

Now, let $s = 2$, i.e. $G-z$ has two connected components H_1, H_2. To prove that G is traceable it suffices to show that each graph H_i is Hamiltonian or at least it contains a Hamiltonian path starting with a neighbour of z. Denote $|H_i| = n_i, i = 1, 2$, and assume $n_1 \leq n_2$. For every two non-adjacent vertices $x, y \in V(H_1)$ we have $d_{H_1}(x) + d_{H_1}(y) \geq n - 5$ since $\sigma_2(G) \geq n - 3$. As $n = n_1 + n_2 + 1$ and $n_1 \leq n_2$, we have $\sigma_2(H_1) \geq 2n_1 - 4$. Hence, for $n \geq 4$ the graph H_1 is Hamiltonian due to well-known Ore's condition for hamiltonicity. If $n \leq 3$, then it is not difficult to verify that G contains a path from z through all vertices of H_1 because $s = 2$. If $n_1 = n_2$, then the same holds for H_2. Then suppose $n_1 < n_2$. For each $x \in V(H_2)$ and $y \in V(H_1)$, we have $d_{H_2}(x) \geq \sigma_2(G) - d_G(y) - 1 \geq n - 3 - n_1 - 1 = n_2 - 3$. Then H_2 is Hamiltonian whenever $n_2 \geq 3$ by the well-known Dirac condition. If $n_2 = 2$, then it is easy to check that G is a traceable graph of order four. □

The following three propositions directly follow from the proof of Theorem 14, and they might be of interest as such. The first two of them can be viewed as extensions of Ore's Theorem 7.

Proposition 15 *If G is a graph of order $n \geq 8$ with connectivity $\kappa(G) = 1$ and $\sigma_2(G) \geq n - 3$, then G is traceable.* □

Proposition 16 *If G is a 2-connected graph of even order n with $\sigma_2(G) \geq n - 3$, $c(G) \geq n - 2$ and $\alpha(G) \leq \frac{n}{2}$, then G is traceable.* □

Proposition 17 *If G is a 2-connected graph of odd order n such that $\sigma_2(G) \geq n-3$ and $c(G) \geq n-2$, then G is on-line AP.* □

Since connectivity and the inequality $\alpha(G) \leq \lceil \frac{n}{2} \rceil$ are necessary conditions for a graph to be AP, Theorem 8 and Theorem 14 immediately imply the following result.

Theorem 18 *If G is a graph of order n with $\sigma_2(G) \geq n-3$, then G is AP if and only if G is on-line AP.* □

3 Erdős-Gallai-type condition

The main result of this section is the following theorem.

Theorem 19 *If G is a connected graph of order, n with either $n \leq 7$ or $n \geq 15$, and size*

$$\|G\| > \binom{n-3}{2} + 6,$$

then G is traceable unless G is a spanning subgraph of the graph $\overline{K_2} \vee K_1 \vee K_{n-3}$ depicted in Figure 5.

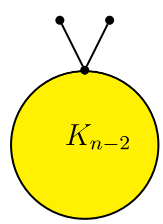

Figure 5: The spanned supergraph $\overline{K_2} \vee K_1 \vee K_{n-3}$ of every non-traceable graph G of order $n \geq 15$ and size $\|G\| > \binom{n-3}{2} + 6$

The initial part of the proof of Theorem 19 is similar to that of the proof of Theorem 10 in [15]. We shall make use of two propositions proved therein, which are consequences of theorems of Woodall [21] and Erdős [8], respectively.

Proposition 20 [15] *For any positive integer δ, if $n = |G|$ and*

$$\|G\| > \binom{n-\delta}{2} + \binom{\delta+1}{2},$$

then $c(G) > n - \delta$. $\square$

For another proposition, define

$$g(n,\delta) = \max\left\{ \binom{n-\delta-1}{2} + \delta(\delta+1), \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \left\lfloor\frac{n}{2}\right\rfloor^2 - n \right\}.$$

Proposition 21 [15] *Let G be a graph of order n and with minimum degree δ. If $\delta \geq \frac{n-1}{2}$ or $\|G\| > g(n,\delta)$, then G is traceable.*

Proof of Theorem 19. Suppose $\|G\| > \binom{n-3}{2} + 6$ and $\delta = \delta(G)$. It follows from Proposition 21 that G is traceable if $\delta \geq \frac{n-1}{2}$ or $g(n,\delta) \leq g(n,3)$. Observe that $\binom{n-\delta-1}{2} + \delta(\delta+1)$ is a quadratic polynomial with respect to δ, so the latter inequality holds whenever $g(n,\delta) \geq \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \lfloor\frac{n}{2}\rfloor^2 - n$. This is the case when $\delta \geq 2$ and

$$\binom{n-3}{2} + 6 \geq \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \left\lfloor\frac{n}{2}\right\rfloor^2 - n.$$

We solve this inequality regarding to the parity of the order n of G. If n is even, then it is equivalent to $n^2 - 22n + 96 \leq 0$, so it holds if $n \geq 16$ or $n \leq 6$. If n is odd, then we have $n^2 - 20n + 91 \leq 0$, and this holds for $n \geq 13$ and for $n \leq 7$.

It follows from Proposition 20 that $c(G) \geq n - 2$. Thus, to complete the proof, we may assume that $c(G) = n - 2$ and $\delta = 1$.

Let C be a longest cycle in G with a fixed orientation, and let u, v be the vertices outside C. If $uv \in E$, then G is traceable. Otherwise, we may assume without loss of generality that $d(v) = 1$ and $d(u) = k \geq 1$. Let $N(u) = \{u_1, \ldots, u_k\} \subseteq V(C)$ and $X = \{u_i^+ : u_i \in N(u)\}$. Clearly, $1 \leq k \leq \lfloor\frac{n-2}{2}\rfloor$ and X is independent as C is a longest cycle. It was proved in [15] that with our assumptions

$$\sum_{i=1}^{k} d_C(u_i^+) \leq \frac{k}{2} c(G),$$

where $d_C(u_i^+) = |N(u_i^+) \cap V(C)|$.

Now, let us estimate the number $f(k) = \|\overline{G}\|$ of edges missing in G. Since X is independent, there are $\binom{k}{2}$ missing edges between vertices of X. Due to the above inequality, there are at least $k(c(G) - k) - \frac{k}{2}c(G)$ missing edges between X and $V(C)\backslash X$. Further, $n-2$ edges incident to v and $n-1-k$ edges incident to u cannot appear in G since $d(v) = 1$ and $d(u) = k$. Therefore,

$$f(k) \geq \binom{k}{2} + k(c(G) - k) - \frac{k}{2}c(G) + 2n - k - 3.$$

Setting $c(G) = n - 2$, we obtain the following inequality

$$f(k) \geq \frac{1}{2}[-k^2 + (n-5)k + 4n - 6].$$

The right-hand side of the above inequality is a quadratic function that is increasing for $k \leq \frac{n-5}{2}$ and decreasing $\frac{n-5}{2} \leq k \leq \frac{n-2}{2}$. Note that $\binom{n}{2} - \binom{n-3}{2} - 6 = 3n - 12$, hence $f(k) \leq 3n - 13$ since $\|G\| > \binom{n-3}{2} + 6$. We have $f(2) = 3n - 10 \geq 3n - 13$ and $f(\frac{n-2}{2}) = \frac{1}{8}(n^2 + 6n - 8) > 3n - 13$ because $\frac{1}{8}(n^2 + 6n - 8) - (3n - 13) = \frac{1}{8}(n^2 - 18n + 96) > 0$ for every n. Therefore, the only possibility $k = 1$. We have $f(1) = \frac{5}{2}n - 6$, and $\frac{5}{2}n - 6 \leq 3n - 13$ only if $n \geq 14$. This means that for $k \in \{5, 6, 7\}$ we have $c(G) > n - 2$, i.e. G is traceable. Hence, we may assume that $n \geq 15$.

Consider now the subgraph $G' = G[V(C)]$. Let $x, y \in V(C)$ with $xy \notin E$. Denote $d_{G'}(x) + d_{G'}(y) = d$. Then

$$\|G'\| > \binom{n-3}{2} + 6 - 2 = \frac{1}{2}(n^2 - 7n + 20).$$

On the other hand,

$$\|G'\| \leq \binom{n-4}{2} + d = \frac{1}{2}(n^2 - 9n + 20 + 2d.)$$

Thus $d \geq n = |G'| + 2 > |G'| + 1$. It follows from the well-known Ore's condition that G' is Hamiltonian-connected. Hence, there is a Hamiltonian path in G' from the neighbour of u in G to the neighbour of$v$ in G. And this path can be clearly extended to a Hamiltonian path of G since $N(u) \cap N(v) = \emptyset$. $\square$

Theorem 19 easily implies the following result.

Theorem 22 *Let G be a connected graph of order $n \geq 15$ and size*

$$\|G\| > \binom{n-3}{2} + 6.$$

Then G is on-line AP unless n is even and G is a spanning subgraph of the graph $\overline{K_2} \vee K_1 \vee K_{n-3}$ depicted in Figure 5.

Proof. Suppose a graph G fulfills the assumptions. If G is traceable, then it is on-line AP. We have shown in the proof of Theorem 19 that every graph of order $n \leq 7$ and size $\|G\| > \binom{n-3}{2} + 6$ is traceable. Then suppose G is non-traceable and $n \geq 15$. By Theorem 19, G is a spanning subgraph of the graph $\overline{K_2} \vee K_1 \vee K_{n-3}$. Let $v_1, \ldots, v_{n-2}$ be consecutive vertices of the longest cycle C of G, and let u_1, u_2 be the two pendant vertices of G adjacent to v_{n-2}. If n is even, then clearly, G has no perfect matching, so it is not AP.

Let then n be odd, and let $\tau = (n_1, \ldots, n_k)$ be an admissible sequence for G. Hence at least one element of an admissible sequence is odd. Let n_i be the first such element of τ. For even elements $n_1, \ldots, n_{i-1}$ of τ, we choose subgraphs induced by consecutive vertices of C starting with v_1 (i.e. $V_1 = G[\{v_1, \ldots, v_{n_1}\}]$, and so on). If $n_i = 1$, then we take $V_i = \{u_1\}$, otherwise $n_i \geq 3$ and we take $V_i = G[\{u_1, u_2, v_{n-2}, \ldots, v_{n-n_i+1}\}]$. In both cases, the remaining graph is traceable. Therefore G is on-line AP. $\square$

To see that the assumption $n \notin \{8, \ldots, 14\}$ in Theorem 19 and Theorem 22 is substantial, observe that the graph $G = K_{\frac{n-2}{2}} \vee \overline{K}_{\frac{n+2}{2}}$ of even order n, with $8 \leq n \leq 14$, has no perfect matching as $\alpha(G) = \frac{n}{2} + 1$. Therefore G is not AP, and hence non-traceable, while $\|G\| > \binom{n-3}{2} + 6$. Clearly, any spanning subgraph of G is not AP.

The case $8 \leq n \leq 14$ was investigated by Bednarz et al. [5].

Theorem 23 [5] *Let G be a graph of order $n \in \{8, \ldots, 14\}$ and size*

$$\|G\| > \binom{n-3}{2} + 6.$$

Then G is AP if and only if G is on-line AP.

Clearly, any spanning subgraph of the graph $\overline{K_2} \vee K_1 \vee K_{n-3}$ for even n is not AP for it does not admit a perfect matching. Hence, Theorem 19 and Theorem 23 imply the following result which in turn justifies the second part of Theorem 11.

Theorem 24 *Let G be a graph of order n and size*

$$\|G\| > \binom{n-3}{2} + 6.$$

Then G is AP if and only if G is on-line AP.

The lower bound for the size of G in Theorem 24 can perhaps be improved, but not below $\binom{n-5}{2} + 10$. Indeed, consider a graph G_n of order n and size $\binom{n-5}{2} + 11$ obtained from a caterpillar $\mathrm{Cat}(5, n-5)$ by substituting the pendant paths P_5 and P_{n-5} by cliques K_5 and K_{n-5}, respectively. It it is easy to see that G_n is (on-line) AP if and only if $\mathrm{Cat}(5, n-5)$ is (on-line) AP. Thus by Theorem 1 and Theorem 3, the graph G_n is AP but not on-line AP for every n such that $n > 20$ and $n \not\equiv 0 \pmod 5$.

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